The exact measures of the Sierpiński d-dimensional tetrahedron in connection with a Diophantine nonlinear system

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Abstract

The Sierpiński d-dimensional tetrahedron Δ^d is the generalization of the most known Sierpiński gasket which appears in many fields of mathematics. Considering the sequences of polytopes $\left\{\Delta_n^d\right\}_n$ that generate Δ^d, we find closed formulas for the sum $v_n^{d,k}$ of the measures of the k-dimensional elements of Δ_n^d, deducing the behavior of the sequences $\left\{v_n^{d,k}\right\}_n$. It becomes quite clear that traditional analysis does not have the adequate language and notations to go further, in an easy and manageable way, in the study of the previous sequences and their limit values; contrariwise, by adopting the new computational system for infinities and infinitesimals developed by Y.D. Sergeyev, we achieve precise evaluations for every k-dimensional measure related to each Δ^d, obtaining a set $W = \left\{v_{①}^{d,k}\right\}_{d,k}$ of values expressed in the new system, which leads us to a Diophantine problem in terms of classical number theory.
To solve it, we work with traditional tools from algebra and mathematical analysis. In particular, we define two kinds of equivalence relations on W and we get a detailed description of the partition of various of its subsets together with the exact composition of the corresponding classes of equivalence.
Finally, we also show as the unique Sierpiński tetrahedron for each dimension d, is replaced, if we adopt Sergeyev's framework, by a whole family of infinitely many Sierpiński d-dimensional tetrahedrons.

Keywords: Sierpiński d-tetrahedron, Fractals, Geometry of polytopes, Numerical infinities and infinitesimals, Grossone, Number theory, Diophantine nonlinear systems.

1. Introduction

The Sierpiński gasket, also called the Sierpiński triangle, is one of the most known and very popular fractal set; it has the exterior shape of an equilateral triangle, and is subdivided recursively into smaller similar triangles, from which the central ones are removed at each step. It has been originally constructed as a curve by the Polish mathematician Wacław Sierpiński one hundred years ago ([27, 28]), but it appeared as a decorative pattern many centuries before, for example in Italian medieval art (as in the Cosmati mosaics, see [30, pp. 43, 873]) and in particular, in several Roman churches and Basiliche from the 11th century (see [7]).
From the original work of Sierpiński, what is known as the *Sierpiński arrowhead curve* is a continuous map of the line segment $[0, 1] \subset \mathbb{R}$ whose image is a fractal curve identical to the Sierpiński gasket; but there are dozens of other different ways to build the Sierpiński gasket and dozens of contexts, in mathematics and other disciplines, where it comes out and finds a variety of applications. For example, it arises from cellular automata (elementary cellular automaton rule 60, 90, 102 and many similar others; see [30] for their extensive descriptions), from chaos games and chaos theory, from puzzle graphs ([26]), from Pascal's triangle, etc., and it has many uses and applications ranging

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from engineering and technology like fractal antennas (see [2]), to programming and computer science, and also, for instance, even in music ([13]).

In this paper we study the generalization of the Sierpiński triangle in each dimension $d \geqslant 2$: the construction of this fractal is well know for every d (for a comprehensive introduction see for example [20], whose first part is entirely devoted to such fractal in dimension 2, 3 and Section 3.8 to general d, or the paper [24] for connections with matrices, digits and "inner products") together with some of its main characteristics as the fractal dimension (see [19]). Although different sources deal with the general d-dimensional case, they are, however, very few in comparison with all the literature on the cases with $d = 2$ and $d = 3$: for instance, the formula obtained in Proposition 2.1, which represents the starting point for the investigations of this paper, seems itself not known before.

The core of the article is, instead, the emergence of a Diophantine nonlinear system with a rather complicated formulation (see (24)) from an unsuspected context as d-dimensional fractal geometry: it is interesting *per se* for number theory, but represents much more than a Diophantine problem because it derives from the application to such fractal of a new computational methodology, recently introduced by Y.D. Sergeyev, which throws new light on the subject. In particular, in contexts similar to ours, this method allow us to consider k-dimensional elements of a fractal object and to determine their *exact measures* as it was an ordinary d-polytope; more generally instead, such a new system allows one to work *numerically* with infinities and infinitesimals in a handy way and, as it is easy to imagine, it is particularly useful in relation to the behavior of models or objects when they are viewed at "infinity".

For detailed introduction surveys on this new numerical system, the reader can see [33, 37, 42, 43] and the book [31] written in a popular way. We inform that this computational methodology has already been successfully applied in optimization and numerical differentiation (see [11, 12, 38, 49]) and in a number of other theoretical and computational research areas such as cellular automata (see [9, 10] and in the context of [3] under investigation), Euclidean and hyperbolic geometry (see [21, 22]), percolation (see [17, 18, 29]), fractals (see [5, 29, 32, 34, 40, 44]), the Riemann zeta function, infinite series and Z-transform (see [6, 35, 39, 48]), the first Hilbert problem, Turing machines and supertasks (see [25, 36, 45, 46]), numerical solution of ordinary differential equations (see [1, 23, 41, 47]), etc.

As regards the structure of the present article and some details on its content, in Section 2 we introduce the Sierpiński tetrahedron Δ^d in any dimension $d \geqslant 2$ together with its generating sequence of d-polytopes $\{\Delta_n^d\}_n$. Then we found, by the above mentioned Proposition 2.1, closed formulas depending on n, for the sum of the measures of the k-dimensional elements of Δ_n^d, and this gives the starting point of the research in this paper. First we deduce the limit measure properties of Δ^d by using classical analysis: this means that, when d and k vary, we obtain a family of elements denoted by $\left\{v_\infty^{d,k}\right\}$, but whose value is zero or $+\infty$ in almost all the cases. Despite they arise from different kinds of measures, in different dimensions, and they carry different meanings and contents, when two of them are both zero or both $+\infty$, they are clearly indistinguishable using the notations of traditional analysis.

In Section 3 instead, we consider the correspondent values of $\left\{v_\infty^{d,k}\right\}$ but using the new computational system: they are denoted by $v_{①}^{d,k}$ and their analysis opens new features and appears from the beginning, full of meaning and rich in interpretations. In addition, contrary to what happens to the values of the family $\left\{v_\infty^{d,k}\right\}$, the elements $v_{①}^{d,k}$ are very often quite different from each other as we will realize in a few simpler cases from (9) and (10), but it is not trivial to say to what point they are really different.

It is also quite evident that some pairs of infinities or infinitesimals arising from (9) and (10), are much more similar than others and this makes us want a sort of classification that emphasizes affinities and relationships. For this purpose, then, we define two equivalence relations between the elements $v_{①}^{d,k}$ (and in general among the numbers of the new computational system, see Definition 3.1), one stronger that the other, and we give by Theorem 3.1, the partition into equivalence classes of the set of the elements $v_{①}^{d,k}$, together with complete information as the number of classes, a set of "minimal" representatives, etc. In particular, in Theorem 3.1 (ii) we obtain that the equivalence classes relative to the stronger relation are all constituted of a single element, and, as consequence, we conclude that the elements of the new system $v_{①}^{d,k}$ are all distinct unlike the traditional case (see Corollary 3.1). Another consequence is that the Diophantine nonlinear system (24) we mentioned earlier, has no nontrivial integer solutions (Corollary 3.2).

Section 4 changes the way of arguing and shows as the unique Sierpiński tetrahedron in dimension d, is replaced, if we use the new computational system, by a whole family of infinitely many Sierpiński d-dimensional tetrahedrons, that give rise to a still larger family of related values $v_{r,①}^{d,k}$. We end the section by suggesting some further directions of research and how it is possible to generalize the results obtained in the previous sections.

Finally, Section 5 is devoted to the conclusions.

Regarding the notations, we advice that, as usual, we will write indifferently $\{a_n\}_n$, $\{a_n\}$, or sometimes simply a_n, to denote a sequence. Moreover, the symbol $\mathbb{N}$ denotes the set of positive integers for us, whilst $\mathbb{N}_0$ includes also zero.

2. The high-dimensional Sierpiński tetrahedron

The generalization of a triangle or tetrahedron to arbitrary dimension $d \geqslant 0$, is what is called a *d-simplex* or, less commonly, a *d-dimensional tetrahedron*; in the following we will often say just a *d-tetrahedron*, for short. More precisely, a d-simplex is a d-dimensional polytope which is the convex hull of $d+1$ affinely independent points in $\mathbb{R}^D$ ($D \geqslant d$) and, of course, it is called *regular* if all its edges have the same length.
The use of simplexes is widespread in many areas of mathematics like algebraic geometry, algebraic topology and especially in singular homology; but we advise the reader that what in literature is called the *standard d-simplex* is different from our notion of *unitary d-simplex* or *unitary d-tetrahedron*, denoted by Δ_0^d and widely used in the following. In fact, whilst the first has edge length $\sqrt{2}$ because it is the convex hull of the standard basis $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ of $\mathbb{R}^{d+1}$, that is

$$\left\{(x_0, \dots, x_d) \in \mathbb{R}^{d+1} \,\middle|\, \sum_{i=0}^{d} x_i = 1 \ \text{ and } \ x_i \geqslant 0 \text{ for all } i = 0, \dots, d\right\},$$

the second, Δ_0^d, is a d-tetrahedron whose edges have all unitary length. More in general, if we denote by $\Delta_0^d(l)$ a regular d-tetrahedron with sides lengths equal to $l \in R_0^+$, we recall from elementary geometry that its d-volume is given by

$$\mathrm{Vol}_d\left(\Delta_0^d(l)\right) = \frac{\sqrt{d+1}}{d!\,\sqrt{2^d}} \cdot l^d. \tag{1}$$

Moreover, if $0 \leqslant k \leqslant d$, every k-dimensional face (briefly *k-face*) of a d-simplex is a k-simplex itself, and since any $k+1$ points from the $d+1$ vertices of a d-simplex identify uniquely a k-face, then, the number $f^d(k)$ of the k-faces of a d-simplex is given by the binomial coefficient

$$f^d(k) = \binom{d+1}{k+1} \tag{2}$$

(for the preceding and other similar properties see, for example, [4, 8, 16]).

The Sierpiński d-dimensional tetrahedron is one of the most simple fractals in dimension d, and it is well known, but not so studied in concrete applications, of course, as its much more common two- and three-dimensional versions, often called the *Sierpiński gasket, sieve, triangle* and the *Sierpiński pyramid, tetrix*, respectively.
We denote the Sierpiński d-tetrahedron by Δ^d, and we sketch briefly, in the following, the easiest way to obtain it by a sequence of d-dimensional polytopes.
Let $d \geqslant 2$ be a fixed integer; to construct such a sequence $\{\Delta_n^d\}_n$ we start with the unitary d-simplex Δ_0^d we talked above. Then we define Δ_1^d as the union of $d+1$ regular tetrahedra of side $1/2$, each one built in a corner of Δ_0^d and continue iteratively the process. More precisely, we pose by convenience

$$l_n := (1/2)^n, \qquad \text{for all } n \in \mathbb{N}_0, \tag{3}$$

and we can adopt two equivalent inductive constructions at the generic step $n \geqslant 2$ to obtain Δ_n^d: the first one consists to repeat a copy of Δ_1^d, scaled by $l_{n-1} = (1/2)^{n-1}$, in each small tetrahedron of side l_{n-1} constituting Δ_{n-1}^d, instead the second construction consists to replicate a copy of Δ_{n-1}^d, scaled by $l_1 = 1/2$, in each of the $d+1$ tetrahedrons of side l_1 composing Δ_1^d.
Finally, for every $d \geqslant 2$, the *d-dimensional Sierpiński tetrahedron* Δ^d is defined as the limit of Δ_n^d for n approaching $+\infty$, i.e. there exists a unique compact set Δ^d which is the limit of the compact sets Δ_n^d. Moreover note that Δ^d is also equal to the intersection $\bigcap_{n\in\mathbb{N}} \Delta_n^d$.
There are several other ways to construct the Sierpiński tetrahedron in dimension d; for example, for a different approach the interested reader can consult [20].

Now we want to attach to each of such fractals Δ^d, some sequences of real numbers $\{v_n^{d,k}\}_n$ which give a k-dimensional valuation of the elements of the generating sequence $\{\Delta_n^d\}_n$. More precisely we pose the following

Definition 2.1. For all integers $d \geqslant 2$ and $n \geqslant 0$, let $v_n^{d,d}$ be the d-volume of Δ_n^d.
Moreover, if $0 \leqslant k < d$, let $v_n^{d,k}$ be the sum of the k-volumes of the *k-dimensional elements* (briefly *k-elements*) lying on the $(d-1)$-dimensional boundary surface of Δ_n^d.

As examples of the previous definitions, note that, if $d = 3$ and $k = 2$, $v_n^{3,2}$ is the surface area of Δ_n^3, instead, if $k = 1$, $v_n^{3,1}$ is the sum of the 1-volumes of the 1-dimensional elements on the surface of Δ_n^3, that is, the sum of the length of its edges. Proposition 2.1 below gives a general expression for $v_n^{d,k}$, but first note that, if $N_n^{(d)}$ is the number of tetrahedra of side l_n which make up Δ_n^d, then

$$N_n^{(d)} = (d+1)^n, \tag{4}$$

for every $n \geqslant 0$.

Proposition 2.1. *For all $n \geqslant 0$ and $d \geqslant 2$, we have*

$$v_n^{d,k} = \begin{cases} \dfrac{\sqrt{k+1}}{k!\,\sqrt{2^k}} \cdot \dbinom{d+1}{k+1} \cdot \left(\dfrac{d+1}{2^k}\right)^n & \text{if } \ 1 \leqslant k \leqslant d, \\[2ex] \dfrac{(d+1)^{n+1} + d + 1}{2} & \text{if } \ k = 0. \end{cases} \tag{5}$$

Proof. If $1 \leqslant k \leqslant d$, then $v_n^{d,k}$ can be computed multiplying the number of d-tetrahedra building Δ_n^d, by the number of k-faces of a single d-tetrahedron, by the k-volume of a regular k-tetrahedron of length side l_n; in symbols, this gives (see (1), (2), (3) and (4))

$$v_n^{d,k} = N_n^{(d)} \cdot f^d(k) \cdot \mathrm{Vol}_k\left(\Delta_0^k(l_n)\right) = \frac{\sqrt{k+1}}{k!\,\sqrt{2^k}} \cdot \binom{d+1}{k+1} \cdot \left(\frac{d+1}{2^k}\right)^n,$$

and we have shown the upper part of (5).
To prove the second branch of (5), it is easy to determine a recursive formula for $v_n^{d,0}$, and then use it iteratively or by induction to obtain the wanted closed expression. Instead, here we prefer to give a different proof, which is very direct and sharp: we know that Δ_n^d is made up of $N_n^{(d)}$ tetrahedra of side l_n, and note that each of the $d+1$ vertices of such a small tetrahedron, is a connecting point between itself and exactly one other tetrahedron. The only exception is given by the exterior $d+1$ vertices of Δ_n^d, that is the ones of Δ_0^d. Hence we can write

$$v_n^{d,0} = \left(N_n^{(d)} \cdot (d+1) + d + 1\right) \cdot \frac{1}{2} = \frac{(d+1)^{n+1} + d + 1}{2},$$

for all $d \geqslant 2$. □

For every $d \geqslant 2$ and $0 \leqslant k \leqslant d$, we pose

$$v_\infty^{d,k} := \lim_{n \to +\infty} v_n^{d,k} \tag{6}$$

and denote by $\dim(\Delta^d)$ the fractal dimension of Δ^d. For comprehensive references about the general theory of the dimension of a fractal, the reader can see [14] or [15]; in our particular case, it is simple to prove that

$$\dim\left(\Delta^d\right) = \frac{\ln(d+1)}{\ln 2} = \log_2(d+1), \tag{7}$$

for every $d \geqslant 2$ (see, for example, [19]).

We close this section with few immediate consequences of Proposition 2.1 and some considerations: it is trivial that the sequence $v_n^{d,k}$ converges if and only if $d \leqslant 2^k - 1$, and converges to a nonzero value if and only if $k \geqslant 2$ and $d = 2^k - 1$; in the last case the limit value is

$$v_\infty^{2^k-1,k} = \frac{\sqrt{k+1}}{k!\,\sqrt{2^k}} \cdot \binom{2^k}{k+1}. \tag{8}$$

Note, moreover, that we obtain the following interesting corollary.

Corollary 2.1. *Let $d \geqslant 2$ be a fixed integer. Then, there exists a nonnegative integer $k \leqslant d$ such that $v_\infty^{d,k}$ is finite and nonzero if and only if the fractal dimension of Δ^d is an integer.*

It seems to be an intriguing and fascinating tool to investigate for which fractals, arising as limit of a sequence of d-dimensional polytopes, a statement similar to Corollary 2.1 holds.

3. The exact measures of the d-dimensional Sierpiński tetrahedron through infinite and infinitesimal computation

In the previous section we computed $v_n^{d,k}$ for finite values of n and we obtained that $v_\infty^{d,k}$ is zero, or a positive number equal to (8), or $+\infty$: that is all what the notations of classical analysis can express. It is also quite obvious that the various zeros and infinities emerging in this way, have not the same meaning, because they arise from a computation of a k-dimensional volume related to a d-dimensional object where k and d are different from case to case. For example, $v_n^{2,0}$, representing the infinite grow of the number of vertices of Δ_n^2 (the generating sequence of the Sierpiński gasket) has a completely different meaning from $v_n^{2,1}$ representing the grow of its perimeter, from $v_n^{4,2}$ which comes from a 2-dimensional area, or from $v_n^{72,6}$ which is related to a 6-volume in a 72-dimensional space, etc. But traditional analysis, because of his language and notations, fails to highlight these differences in a simple and computationally efficient way, and such infinite quantities are all written likewise by using the same symbol $+\infty$.
An analogous discussion applies also to zeros arising from the limit (6), and several other evidences provided in the papers mentioned above, reinforce the reflection of how ordinary computational systems are, sometimes, not so convenient to describe effectively and to treat many phenomena occurring in contemporary mathematics, like those we are speaking about.

From here on, we assume that the reader is familiar with the new computational method developed by Y.D. Sergeyev, as explained in the Introduction; we recall that one can easily consult [31, 33, 37, 42, 43] for detailed surveys and introductive essays on the subject. In this section we show as, adopting this new computing system for infinities and infinitesimals quantities based on the new entity ① called *grossone*, we can give a rich description of the behavior at infinity of the previous constructive processes as, and maybe more than, the one at finite.
We begin by noting that if we execute ① steps in the construction of Δ^d, we obtain the following values for the related k-volumes

$$v_{①}^{d,k} = \frac{\sqrt{k+1}}{k!\,\sqrt{2^k}}\binom{d+1}{k+1}\cdot\left(\frac{d+1}{2^k}\right)^{①} \tag{9}$$

in the case $1 \leqslant k \leqslant d$, and

$$v_{①}^{d,0} = \frac{d+1}{2}\cdot(d+1)^{①} + \frac{d+1}{2} \tag{10}$$

for $k = 0$ (see Proposition 2.1). Obviously, if we carry out a different (infinite) number of steps like $N = ① - 4$ or $N = 2① + 3$, then we obtain different fractals (that we be can denoted, for example, by $\Delta^d_{①-4}$ and $\Delta^d_{2①+3}$, respectively, to distinguish them from $\Delta^d = \Delta^d_{①}$) and consequently this yields different related values $v^{d,k}_{①-4}$ and $v^{d,k}_{2①+3}$ in the place of (9) and (10). However, for the sake of simplicity and to not weigh down the notations too much, we will do most of the considerations and investigations in this article limiting ourselves to the $N = ①$ case (see also Section 4 for further discussions and suggestions of research on much more general cases).

Now, note for example, that none of the expressions in (9) and (10) is longer zero as so many limit values in (6). To understand better the extent of change obtained by using the new system, we think it could be useful for the reader, to write the first few explicit values of $v_\infty^{d,k}$, and compare them with the correspondent ones of $v_{①}^{d,k}$.
As he can easily see by direct evaluation, the first few elements obtained from (9) and (10) are actually distinct, but this does not mean that, continuing to compute the subsequent elements of the family $\{v_{①}^{d,k}\}$, they will be all distinct one from the others. This is rather interesting: although we use a powerful numerical system that allows a great differentiation for infinite and infinitesimal values, it may still happen that two values remain indistinguishable, and the occurrence of such a thing is certainly worthy of note and of further attentions on the underlying geometry producing this phenomenon.

At this point it should be clear that there is no reason to decide *a priori* a case like the one just exposed, but to further convince the reader and to stimulate a deeper reflection on the problem, let us consider also the following example.

Example 3.1. Let $l \in \mathbb{R}^+$ and d any integer greater than one. We already defined, just before equation (1), the symbol $\Delta_0^d(l)$: starting from the last, rather than $\Delta_0^d = \Delta_0^d(1)$, we can construct, by a complete analogy, a sequence of polytopes $\{\Delta_n^d(l)\}_n$, a fractal $\Delta^d(l)$ scaled by a factor l with respect Δ^d, and for every integer $k \in \{0, 1, \dots, d\}$ we can attach a sequence of real numbers $\{v_n^{d,k}(l)\}_n$ and a value $v_{①}^{d,k}(l)$ expressed in the new system. It is easy to rewrite a generalization of Proposition 2.1 just by noting that

$$v_n^{d,k}(l) = v_n^{d,k} \cdot l^k$$

for all integers $0 \leqslant k \leqslant d$, and consequently

$$v_{①}^{d,k}(l) = v_{①}^{d,k} \cdot l^k.$$

Now, by a short computation it is easy to see that, taking for instance $l = \sqrt{3}/7$ instead of $l = 1$, we find the following coincidence of values

$$v_{①}^{3,1}\left(\frac{\sqrt{3}}{7}\right) = v_{①}^{7,2}\left(\frac{\sqrt{3}}{7}\right), \tag{11}$$

and if we pose $l = \sqrt{3}/5$, we have

$$v_{①}^{2,1}\left(\frac{\sqrt{3}}{5}\right) = v_{①}^{5,2}\left(\frac{\sqrt{3}}{5}\right). \tag{12}$$

It means that in a family analogue to $\{v_{①}^{d,k}\}$, obtained by taking $l = \sqrt{3}/7$ instead of $l = 1$, we have at least two coincident values (the ones appearing in (11)), and the same happens by taking $l = \sqrt{3}/5$ as well (the ones in (12)).

Of course we can extend the previous example with many other cases because the value of l is computed intentionally to have a convergence of at least two values of $v_{①}^{d,k}(l)$. But we inform the reader that we have some examples arising from d-dimensional fractals more complex than Δ^d, in which many values (sometimes infinitely many) of the kind $v_{①}^{d,k}$ all coincide spontaneously to a single numerical ①-based expression: it is not possible to discuss them here, but probably, they will appear with some details in a successive paper. In any case, for our purposes, it is sufficient to consider what happens in (11) and (12).

In conclusion, there are now sufficient reasons to try to understand better which relations exist among the elements $v_{①}^{d,k}$: not only if some of them coincide, but it is also interesting to investigate which of them are "closer" than others, which are infinities or infinitesimal of a "similar kind", etc. For instance, it is immediate to realize that $v_{①}^{2,2}$ is an infinitesimal with an expression that "looks like" $v_{①}^{5,3}$, or $v_{①}^{3,3}$ is another that "looks like" $v_{①}^{7,4}$; but the last comes from a 4-volume of 7-dimensional object so, before using new notations, it is difficult to see analogies with $v_{①}^{3,3}$ which is the ordinary 3-volume of a (fractal) object in the usual 3-dimensional space $\mathbb{R}^3$.
The same happens for infinities, and maybe, it is also more evident: $v_{①}^{3,0}$ is an infinite that "looks like" $v_{①}^{7,1}$, but the first comes from counting a set of points, i.e. the vertices of an object in $\mathbb{R}^3$, while the second arises from the measure of a 1-dimensional length related to an object in $\mathbb{R}^7$.
We remark how the last as the previous ones, although being simple examples, are rather hidden if no notations and new methods are used to highlight these similarities.

We begin our analysis on the values $v_{①}^{d,k}$ by posing the following general definition.

Definition 3.1. Let α and β be any nonzero quantities expressed in the new computational system.

(i) α and β are said *of the same order* (in symbols $\mathrm{ord}(\alpha) = \mathrm{ord}(\beta)$, or $\alpha \sim_{\mathrm{ord}} \beta$) if their quotient is finite (positive or negative) but not infinitesimal. That is, if there exist two finite natural numbers $n, m \in \mathbb{N}$ such that $0 < 1/n < |\alpha/\beta| < m$.
In case α, β are both infinite or infinitesimal quantities, they are also called *infinities of the same order* or *infinitesimals of the same order*, respectively.

(ii) α and β are said *equivalent* (in symbols $\alpha \sim_{eq} \beta$, or simply $\alpha \sim \beta$) if their quotient is 1 up to infinitesimals. This means that $1 - 1/n < \alpha/\beta < 1 + 1/n$ for all finite $n \in \mathbb{N}$.
As before, in case α, β are infinite or infinitesimal quantities, they are also called *equivalent infinities* or *equivalent infinitesimals*, respectively.

Example 3.2. To give some examples on the previous definitions, for the purposes of this paper, we can just consider elements with very simple expressions, or at most of the kind of the $v_{①}^{d,k}$.

(1) $①^2 + \frac{1}{3}① - 3$ is an infinite not of the same order of ①, obviously.

(2) $5 - 2①^{-3}$ and $-2+①^{-1}$ are two finite elements with the same order, but not equivalent.

(3) $\dfrac{1}{3①^2 - ①}$ and $\dfrac{2}{2①^2 - 3 + 5①^{-2}}$ are two infinitesimals of the same order of $①^{-2}$, but only the second is equivalent to it as well.

(4) If d is any integer $\geqslant 2$, then $(d+1)^{①+5}$ has the same order of $v_{①}^{d,0}$, but they are not equivalent (see (10)); instead $(1/2) \cdot (d+1)^{①+1}$ is an infinity equivalent to $v_{①}^{d,0}$, but they are not equal numbers because they differ by a finite quantity.

(5) Assume that $\alpha = a_m\, ①^m + a_{m-1}\, ①^{m-1} + \ldots + a_s\, ①^s$ and $\beta = b_n\, ①^n + b_{n-1}\, ①^{n-1} + \ldots + b_t\, ①^t$, where $m \geqslant s$, $n \geqslant t$ are integers, and a_i, b_j are real numbers with the leading coefficients a_m, b_n different from zero.
Then $\operatorname{ord}(\alpha) = \operatorname{ord}(\beta)$ if and only if $m = n$ and, in particular, they are infinities of the same order if $m = n > 0$ or infinitesimals of the same order if $m = n < 0$. Moreover, α and β are equivalent if and only if $m = n$ and $a_m = b_n$. (Obviously, it is enough to note just that $\alpha \sim_{eq} a_m\, ①^m$ and $\beta \sim_{eq} b_n\, ①^n$.)

The interested reader can find less trivial examples by himself, also using references like [5] in which are discussed some power series expansions in infinitesimal elements of the new system, which arise from a geometric context of space-filling curves.

It is trivial that both the relations introduced in Definition 3.1 (i) and (ii) are equivalence relations, so for example, we can ask about the equivalence class of $v_{①}^{d,k}$, using the relation $\sim_{\text{ord}}$ or $\sim_{eq}$.
We will soon see as Corollary 3.1, which is an immediate consequence of Theorem 3.1 (ii), states that the elements $v_{①}^{d,k}$ are actually all distinct; but Theorem 3.1 establishes much more. In fact, our purpose is to give information about the partition of the following sets

$$V^d := \left\{ v_{①}^{d,k} \,\middle|\, 0 \leqslant k \leqslant d \right\}, \quad W^d := \bigcup_{i=2}^{d} V^i, \quad \text{for all } d \geqslant 2, \qquad \text{and} \qquad W := \bigcup_{d \geqslant 2} V^d, \tag{13}$$

in equivalence classes. We denote the equivalence class of $v_{①}^{d,k}$ in W, with respect the relation $\sim_{\text{ord}}$ or $\sim_{eq}$, by $\left[v_{①}^{d,k}\right]_{\text{ord}}$ and $\left[v_{①}^{d,k}\right]_{eq}$, respectively.
Note, moreover, that we can likewise consider any family $\{v_N^{d,k}\}$, where N is another infinite number other than ①, or also the union of many such families, but this is beyond the scope of the present paper.

Before to state Theorem 3.1, we write in the following lemma an important consequence of the equations (9) and (10).

Lemma 3.1. *For all $d \geqslant 2$ and $0 \leqslant k \leqslant d$ we have* $\operatorname{ord}\left(v_{①}^{d,k}\right) = \operatorname{ord}\left(\left(\dfrac{d+1}{2^k}\right)^{①}\right)$. *Moreover*

$$v_{①}^{t,h} \sim_{\text{ord}} v_{①}^{d,k} \qquad \textit{if and only if} \qquad t + 1 = 2^{h-k} \cdot (d+1). \tag{14}$$

The proof of the previous lemma is an easy check we leave to the reader. So now we are ready to state the main theorem: part (i) concerns the first equivalence relation and part (ii) the second. In particular, part (i) gives full information on the number of equivalence classes existing in W^d for every $d \geqslant 2$ (see (15)), then for every class in W it gives the smallest representative (see (16)) and, finally, starting from one of such representatives, it yields an explicit writing for the other elements belonging to its class (see (17)).

Theorem 3.1.

(i) Let $d \geqslant 2$ and denote by ν_d the number of equivalence classes in the set W^d with respect the equivalence relation $\sim_{\text{ord}}$. Then, if $\lceil a \rceil$ is the ceiling of $a \in \mathbb{R}$, we have

$$\nu_d = \begin{cases} 3 & \text{if } d = 2, \\ \dfrac{3d^2+9d+6}{8} + \dfrac{(-1)^d}{4} \cdot \left\lceil \dfrac{d+1}{2} \right\rceil & \text{if } d \geqslant 3. \end{cases} \tag{15}$$

Moreover, a system of minimal (i.e. with d, k minimal) representatives of the classes in W is

$$\mathcal{R}_W := \bigcup \left\{ V^d \;\middle|\; d \text{ even} \geqslant 2 \text{ or } d = 3 \right\} \cup \left\{ v_{①}^{t,h} \;\middle|\; t \text{ odd} \geqslant 5 \text{ and } h = 0 \text{ or } \frac{t+3}{2} \leqslant h \leqslant t \right\}, \tag{16}$$

and for every $v_{①}^{t,h} \in \mathcal{R}_W$, its equivalence class can be written as

$$\left[v_{①}^{t,h} \right]_{\text{ord}} = \left\{ v_{①}^{2^{j-h}(t+1)-1,\, j} \;\middle|\; j \in \mathbb{N}_0,\ j \geqslant h \right\}. \tag{17}$$

(ii) Each equivalence class in W under the relation $\sim_{eq}$ consists of a single element.

Proof. (i) We begin the proof of (i) by proving the last assertion, but first of all, we want to remark that the word "minimal", of course, does not refer to the non-redundancy of (16) which is implicit in the definition of a system, but it means that the representatives are chosen with d and k minimal.
From condition (14) of Lemma 3.1, it is clear that if two elements of the same set V^d have equal order, then they must to coincide. Now, we think that it is enlightening to write explicitly how some of the first classes of equivalence are composed: recalling the cited lemma, we have

$$\begin{aligned}
\left[v_{①}^{2,0} \right]_{\text{ord}} &= \left\{ v_{①}^{3\cdot 2^h - 1, h} \mid h \in \mathbb{N}_0 \right\} &&= \left\{ v_{①}^{2,0}, v_{①}^{5,1}, v_{①}^{11,2}, v_{①}^{23,3}, \ldots \right\}, \\
\left[v_{①}^{2,1} \right]_{\text{ord}} &= \left\{ v_{①}^{3\cdot 2^{h-1} - 1, h} \mid h \in \mathbb{N} \right\} &&= \left\{ v_{①}^{2,1}, v_{①}^{5,2}, v_{①}^{11,3}, v_{①}^{23,4}, \ldots \right\}, \\
\left[v_{①}^{2,2} \right]_{\text{ord}} &= \left\{ v_{①}^{3\cdot 2^{h-2} - 1, h} \mid h \geqslant 2 \right\} &&= \left\{ v_{①}^{2,2}, v_{①}^{5,3}, v_{①}^{11,4}, v_{①}^{23,5}, \ldots \right\}, \\
\left[v_{①}^{3,0} \right]_{\text{ord}} &= \left\{ v_{①}^{4\cdot 2^h - 1, h} \mid h \in \mathbb{N}_0 \right\} &&= \left\{ v_{①}^{3,0}, v_{①}^{7,1}, v_{①}^{15,2}, v_{①}^{31,3}, \ldots \right\}, \\
\left[v_{①}^{3,1} \right]_{\text{ord}} &= \left\{ v_{①}^{4\cdot 2^{h-1} - 1, h} \mid h \in \mathbb{N} \right\} &&= \left\{ v_{①}^{3,1}, v_{①}^{7,2}, v_{①}^{15,3}, v_{①}^{31,4}, \ldots \right\}, \\
\ldots \quad & \quad\ldots && \quad\ldots
\end{aligned} \tag{18}$$

For t odd $\geqslant 5$, the classes

$$\left[v_{①}^{t,1} \right]_{\text{ord}}, \quad \left[v_{①}^{t,2} \right]_{\text{ord}}, \quad \ldots, \quad \left[v_{①}^{t,(t+1)/2} \right]_{\text{ord}} \tag{19}$$

are the only yet present in W^{t-1}; in fact, it is a simple check left to the reader, to show that for t, h fixed with $0 \leqslant h \leqslant t$ and $2 \leqslant h$, the equation

$$t = (d+1) \cdot 2^{h-k} - 1 \tag{20}$$

is resoluble with respect to d, k such that $0 \leqslant k \leqslant d < t$ and $d \geqslant 2$, if and only if t is an odd integer $\geqslant 5$ and $1 \leqslant h \leqslant (t+1)/2$, as we claimed implicitly in (19). Hence (16) is demonstrated and also (17) is clear.
At this point, to prove (15) we can use an inductive argument on d. If d is equal to 2 or 3, it is trivially true; now, assuming the formula true for $d-1$ we want to verify it for $d \geqslant 4$. From the above discussion we can write

$$\nu_d \;=\; \nu_{d-1} + \left(\frac{3}{4} + (-1)^d \cdot \frac{1}{4}\right) \cdot (d+1) \tag{21}$$

for all $d \geqslant 4$; hence, using the inductive hypothesis and noting that $d + 1 - \lceil d/2 \rceil = \lceil (d+1)/2 \rceil$, we obtain

$$\begin{aligned} \nu_d \;&=\; \frac{3(d-1)^2 + 9(d-1) + 6}{8} + \frac{(-1)^{d-1}}{4} \cdot \left\lceil \frac{d}{2} \right\rceil \;+\; \frac{3}{4}(d+1) + (-1)^d \cdot \frac{1}{4}(d+1) \\ &=\; \frac{3d^2 + 9d + 6}{8} \;+\; \frac{(-1)^d}{4} \cdot \left\lceil \frac{d+1}{2} \right\rceil, \end{aligned}$$

and the inductive step is proved.

(ii) Let d any integer $\geqslant 2$ and consider the equivalence class $\left[v_{①}^{d,k}\right]_{\mathrm{ord}}$; to prove (ii) it is sufficient to show that any two elements in $\left[v_{①}^{d,k}\right]_{\mathrm{ord}}$ are not equivalent according to Definition 3.1 (ii).
If $k \geqslant 1$, from (17) and (9), we can write the class of $v_{①}^{d,k}$ as follows

$$\left[v_{①}^{d,k}\right]_{\mathrm{ord}} = \left\{ v_{①}^{2^{h-k}(d+1)-1,\,h} \;\middle|\; h \geqslant k \right\} = \left\{ \frac{\sqrt{h+1}}{h!\,\sqrt{2^h}} \binom{2^{h-k}(d+1)}{h+1} \cdot \left(\frac{d+1}{2^k}\right)^{①} \;\middle|\; h \geqslant k \right\}, \tag{22}$$

instead, if $k = 0$, from (9) and (10), we have

$$\begin{aligned} \left[v_{①}^{d,0}\right]_{\mathrm{ord}} &= \left\{v_{①}^{d,0}\right\} \cup \left\{ v_{①}^{2^h(d+1)-1,\,h} \;\middle|\; h \geqslant 1 \right\} \\ &= \left\{ \frac{d+1}{2} \cdot (d+1)^{①} + \frac{d+1}{2} \right\} \cup \left\{ \frac{\sqrt{h+1}}{h!\,\sqrt{2^h}} \binom{2^h(d+1)}{h+1} \cdot (d+1)^{①} \;\middle|\; h \geqslant 1 \right\}. \end{aligned} \tag{23}$$

Now, pose $x = h - k$ and consider the family of sequences $a_{d,k} : \mathbb{N}_0 \to \mathbb{R}$ depending from the parameters d, k, and defined by

$$a_{d,k}(x) := \frac{\sqrt{k+1+x}}{(k+x)!\,\sqrt{2^{k+x}}} \binom{2^x(d+1)}{k+1+x}.$$

Note that $a_{d,k}(x)$ assumes all the values appearing as coefficient of $\left((d+1)/2^k\right)^{①}$ in (22) and (23), with the exception of $(d+1)/2$ which is the coefficient of $(d+1)^{①}$ in the element in the first set of (23); since $a_{d,0}(0) = d+1 > (d+1)/2$, to prove our thesis, it is sufficient to show that the sequences $a_{d,k}(x)$ are strictly increasing as functions of $x \in \mathbb{N}_0$, for any choice of the parameters d, k.

To do this, we can study the function $a_{d,k}(x)$ by extending, as usual, the discrete factorial to complex numbers and by using the well known *gamma function* $\Gamma(x)$, *beta function* $B(x)$ and *digamma function* $\psi(x)$ with their properties. But here we want to give a more elementary proof which uses only techniques of basic calculus.
Note hence that, for all d, k, x, we have

$$\begin{aligned} \frac{a_{d,k}(x+1)}{a_{d,k}(x)} \;&=\; \sqrt{\frac{k+2+x}{k+1+x}} \cdot \frac{1}{\sqrt{2}\,(k+1+x)} \cdot \binom{2^{x+1}\,(d+1)}{k+2+x} \cdot \binom{2^x\,(d+1)}{k+1+x}^{-1} \\ &>\; \frac{1}{\sqrt{2}\,(k+1+x)} \cdot \binom{2^{x+1}(d+1)}{k+2+x} \cdot \binom{2^x\,(d+1)}{k+1+x}^{-1} \end{aligned}$$

$$> \frac{1}{\sqrt{2}\,(k+1+x)} \cdot \frac{\prod_{i=0}^{k+x} 2^{x+1}(d+1)-i}{\prod_{i=0}^{k+x} 2^{x}(d+1)-i} \cdot \frac{2^{x+1}(d+1)-(k+1+x)}{k+2+x}$$

$$> \frac{1}{\sqrt{2}\,(k+1+x)} \cdot 2^{k+1+x} \cdot \frac{2^{x+1}(d+1)-(k+1+x)}{k+2+x},$$

and, if we define the function $\varphi_{d,k}$ by

$$\varphi_{d,k}(x) := \frac{2^{k+1+x} \cdot \left(2^{x+1}(d+1)-(k+1+x)\right)}{\sqrt{2}\,(k+1+x)(k+2+x)},$$

it is sufficient to show that $\varphi_{d,k}(x) > 1$ for every d, k, and any $x \in \mathbb{N}_0$. We leave this check to the reader, suggesting him to treat previously the cases $\varphi_{d,k}(0)$, $\varphi_{2,2}(1)$, $\varphi_{3,3}(1)$ and $\varphi_{2,2}(2)$, and to use the following inequality otherwise

$$\varphi_{d,k}(x) \geqslant \frac{(k+2+x)\cdot((x+2)(d+1)-(k+1+x))}{\sqrt{2}\,(k+1+x)(k+2+x)} = \frac{dx+2d-k+1}{\sqrt{2}\,(k+1+x)}.$$

□

Note that, in the preceding theorem, the ①-based system is present only at the notational level; we can avoid completely its use provided that appropriate, but probably heavier, notations are introduced.
As previously announced, a trivial consequence of part (ii) of Theorem 3.1 is the following

Corollary 3.1. *The elements $v_{①}^{d,k}$ are all distinct for every $d \geqslant 2$ and $0 \leqslant k \leqslant d$.*

The problem of deciding whether there is coincidence of values in the set W, can also be expressed in terms of pure number theory: in the case $k, h \geqslant 1$, it is in fact equivalent to find nontrivial integer solutions of the following nonlinear Diophantine system

$$\begin{cases} \dfrac{\sqrt{k+1}}{k!\,\sqrt{2^k}} \cdot \dbinom{d+1}{k+1} = \dfrac{\sqrt{h+1}}{h!\,\sqrt{2^h}} \cdot \dbinom{t+1}{h+1} \\ \dfrac{d+1}{2^k} = \dfrac{t+1}{2^h} \end{cases} . \tag{24}$$

But to prove the nonexistence of such solutions of a Diophantine problem like (24), is a non trivial issue; for example, by using the most powerful computer algebra systems or scientific computational software available today like, for instance, *Mathematica*® 11.0 by Wolfram Research Inc. or many others, it is not possible to obtain any answer except for very small values of d and t cause the complexity of (24). Instead, as trivial consequence of Theorem 3.1 (ii), we can more precisely state

Corollary 3.2. *There are no integer solutions $(d, t, k, h) \in \mathbb{N}^4$ of the system* (24)*, such that $1 \leqslant k \leqslant d$, $1 \leqslant h \leqslant t$ and $2 \leqslant d < t$.*

Problems like the one solved by the previous corollary, are interesting by themselves for number theory, but it is difficult to pose such problems independently and to write a system which is interesting to study like (24), if it does not arise from some external source as, in this case, from fractal geometry and especially from the use of new notations offered by Sergeyev's system. This in particular means that it is possible to apply the example of this paper to many other geometrical entities and in many other contexts, generating a great deal of new problems.
Note moreover that not only the conception of a so complex system like (24), but also the method of proof used by us in Theorem 3.1, is inspired by geometry and by the notations from the new system. In fact, one who does not know

the geometric meaning of the problem (24), to solve it, would probably try a totally different way like, for example, that of studying the linear independence over $\mathbb{Q}$ of the two following roots (see the first equation of (24))

$$\sqrt{2^h(k+1)} \qquad \text{and} \qquad \sqrt{2^k(h+1)}\,.$$

Instead, as we have already said, our demonstration method focuses on achieving a classification of the set W that gives us much more information on the fractal geometry of the Sierpiński's d-tetrahedron, than the simple solution of the problem (24). As a further example, note that W is clearly a totally ordered set, but the explicit relation is not obvious at first sight. Another immediate consequence of Lemma 3.1 and of the proof of Theorem 3.1 (ii), is an express rule to order the elements of W, as the following

$$v_{①}^{t,h} \;>\; v_{①}^{d,k} \qquad \text{if and only if} \qquad \begin{cases} t > 2^{h-k}\cdot(d+1)-1 & \text{or} \\ t = 2^{h-k}\cdot(d+1)-1 & \text{and} \quad h>k \end{cases}, \tag{25}$$

where ">" has the obvious meaning also in the new system.

4. A family of infinitely many Sierpiński d-dimensional tetrahedra

From the previous section it is clear the great contribution by the new computational system to innovate and to bring with new lights many traditional tools. We conclude the paper with an important observation which gives one more evidence.

Recalling the construction of Δ^d, it is obvious that if we begin the building process from the simplex Δ_0^d or a few steps ahead, the final result, that is the fractal Δ^d, is the same in both cases; but what is a so trivial fact in traditional analysis is no longer true employing a more precise numerical system which is able to take into account the subtle difference from a number of steps equal, for example, to the elements in $\mathbb{N}$ or in $\mathbb{N}-\{1\}$. In fact, as explained in more details in [34] and [44], the new system distinguishes and considers as different the fractals obtained after ① steps starting from some Δ_r^d or some Δ_s^d (where $r \neq s$), as well as the fractals $\Delta^d_{①+2}$ and $\Delta^d_{①-5}$.
Hence, for each $d \geqslant 2$, there is not only a unique d-dimensional Sierpiński tetrahedron, but we can observe a family of infinitely many d-dimensional Sierpiński tetrahedra.

In this regard, we denote by $\Delta^d_{r,n}$, $r,n \in \mathbb{N}_0$, $d \geqslant 2$, the d-dimensional polytope resulting from n iterations starting from Δ_r^d; the definition of $v_{r,n}^{d,k}$ and $v_{r,①}^{d,k}$ are clearly the same as those of $v_n^{d,k}$ and $v_{①}^{d,k}$ given in the previous sections, but starting from Δ_r^d rather than Δ_0^d. Obviously, looking only at the numerical results relative to finite values of r,n, it is trivial that $v_{r,n}^{d,k} = v_{r+n}^{d,k}$ and there would be no need to introduce new notations in this way; but instead, it is important to write $v_{r,n}^{d,k}$ and $v_{r,①}^{d,k}$ to take separate, in the second right subscript, the number of steps done in the considered construction from the starting configuration and to avoid confusion.
Having now the necessary notations, we can write the generalizations of (9) and (10) for all $r \in \mathbb{N}_0$ as follows

$$v_{r,①}^{d,k} \;=\; \frac{\sqrt{k+1}}{k!\,\sqrt{2^k}}\binom{d+1}{k+1}\left(\frac{d+1}{2^k}\right)^r \cdot \left(\frac{d+1}{2^k}\right)^{①} \tag{26}$$

in the case $1 \leqslant k \leqslant d$, and

$$v_{r,①}^{d,0} \;=\; \frac{(d+1)^{r+1}}{2}\cdot(d+1)^{①} \;+\; \frac{d+1}{2} \tag{27}$$

if $k=0$. Note that we can compute in the same way $v_{r,N}^{d,k}$ when N is an infinite number different from ①, as jet observed after equation (10).
Recall that Definition 3.1 introduced the equivalence relations $\sim_{\text{ord}}$ and $\sim_{eq}$ in full generality, so they determine two partitions of the extended set of numbers $\mathcal{W} := \left\{ v_{r,①}^{d,k} \,\middle|\, d-2, k, r \in \mathbb{N}_0, 0 \leqslant k \leqslant d \right\}$ and, therefore, a more complex version which generalizes Theorem 3.1 would arise, following this line of ideas. But not only; being the elements of $\mathcal{W}$ precise numbers, it is possible to compare in various ways and for several purposes, the values in $\mathcal{W}$, one with the others or some subsets of $\mathcal{W}$ with others.

Going even further, it is also possible, by using some other amount of theory and in particular the way to execute ① steps each time for many consecutive processes as explained in the already mentioned references, to study the set of the elements $v_{r,s}^{d,k}$, when not only s, but also r can be an infinite number (less or equal to s) of the new computational system.

5. Conclusions

After recalling the construction of the well known d-dimensional Sierpiński tetrahedron Δ^d together with its generative sequence of polytopes Δ_n^d, the starting point of the paper was to find explicit closed formulas for the sum $v_n^{d,k}$ of the measures of the k-dimensional elements of Δ_n^d, for all d and k (Proposition 2.1). From them, we easily deduced the behavior of the sequences $\{v_n^{d,k}\}_n$ and in particular the value of their limit $v_\infty^{d,k}$. Using Sergeyev's computational system, we introduced the elements $v_{①}^{d,k}$ and we showed the possibility to operate with the infinite and infinitesimal quantities coming out from the sequences $\{v_n^{d,k}\}_n$ in a handy way, as for finite real values of standard mathematics. Unlike the elements $v_\infty^{d,k}$ that bring very poor information because they are, for the most part, zero or $+\infty$, the elements $v_{①}^{d,k}$ seem at first sight quite different from each other, and carry with them rich information about their original generating sequence. The problem of determining whether all these values are distinct or not, is not trivial and may also have a negative answer (remember Example 3.1). We have seen that in the major case, it is connected to the existence of integer solutions of the nonlinear Diophantine system (24), but instead to study directly this problem, we wanted to understand deeply the similarities, affinities and relationships existing between the elements of the set $W = \left\{v_{①}^{d,k} \,\middle|\, d \geqslant 2 \text{ and } 0 \leqslant k \leqslant d\right\}$. To do this, we introduced by Definition 3.1, two new equivalence relations $\sim_{\text{ord}}$ and $\sim_{eq}$ among the numbers of the new computational system based on the grossone, and they produce two partitions of the set W into equivalence classes $[\ ]_{\text{ord}}$ and $[\ ]_{eq}$.
Theorem 3.1 (i) concerns the relation $\sim_{\text{ord}}$ and gives complete answers about the composition of the relative equivalence classes, about their number in $W^d = \left\{v_{①}^{j,k} \,\middle|\, 2 \leqslant j \geqslant d \text{ and } 0 \leqslant k \leqslant j\right\}$ (denoted by ν_d, see equation (15)) and about a set of minimal representatives (denoted by $\mathcal{R}_W$, see equation (16)). Instead, part (ii) of the same theorem concerns the relation $\sim_{eq}$ and states that every equivalence class $[\ ]_{eq}$ in W consists of a single element. As consequences, this in particular allowed us to claim that all the numbers in W are effectively distinct (Corollary 3.1) and the Diophantine system (24) has no nontrivial solutions (Corollary 3.2).

Lastly, we showed in Section 4 as the unique Sierpiński tetrahedron in dimension d is replaced, if we use the new computational system, by a whole family of infinitely many Sierpiński d-dimensional tetrahedrons. They give rise to an even larger family of related values $\left\{v_{r,①}^{d,k} \,\middle|\, d \geqslant 2,\ 0 \leqslant k \leqslant d,\ r \geqslant 0\right\}$ which generalizes the set W (note that W is in fact obtained for $r = 0$), and on which it is possible to deepen into further directions the researches and the results obtained in this paper.

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