The Cantor-Vitali function and infinity computing

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Abstract. In this work we consider the Cantor-Vitali function $c : [0, 1] \to [0, 1]$, constructed as limit of a sequence of functions $\{f_n\}_{n\in\mathbb{N}_0}$. In particular, we give formulas for the length of the graph of the approximating functions and will discuss them, together with the length of the graph of c, also by using infinity computing.

Keywords: Cantor-Vitali function · Cantor set · Fractals · Grossone · Infinity computing

1 Introduction

The *Cantor-Vitali function* $c : [0, 1] \to [0, 1]$ is a uniformly continuous and surjective function, defined on the closed interval $[0, 1]$ of the real line and having the same image. It is also called the *Cantor ternary function* or *Lebesgue's singular function*, and it has amazing peculiarities: for instance, it is constant on all countable complementary intervals of the Cantor set and yet it is an increasing function on the whole domain despite having derivative zero on all these intervals (see for example [6, 35]).

In the present work we use a definition of c as limit of a convergent sequence of functions $\{f_n(x)\}_{n\in\mathbb{N}_0}$, where $\mathbb{N}_0$ is the set of non-negative integers (see Sect. 2). The area under the f_n's is much less interesting than the length l_n of their graphs for which we give a closed formula depending on n (see (6)).

In Sect. 3 we use the *grossone*-based computational system introduced by Y.D. Sergeyev in the early 2000's: we refer the reader to [30, 33] for detailed introductory surveys on the subject which show how to work numerically with infinite and infinitesimals numbers in a very easy and handy way, or to the book [29] written in a popular way.

With traditional mathematics we can only say that the limit of the lengths l_n is 2 (see (7)). Instead, by using the new system, we can consider a single sequence or a family of chained ones, obtaining, in this way, a whole range of infinitely many distinct values which make the result more refined and accurate. From geometrical considerations we deduce that all these values (obtained after any infinite number of steps) are less than 2 and differ from it by infinitesimal quantity expressed through the new computational method in a precise and sharp way. Easy examples are explicitly given in (11) and (12).

2 The approximating functions f_n and some length formulas

The functions

$$f_n : [0,1] \longrightarrow [0,1]$$

are recursively defined for all integer $n \geq 0$ as follows:

$$f_0 := \mathrm{id}_{[0,1]},$$

is the identity on the closed interval $[0,1]$, i.e. $f_0(x) = x$ for all $x \in [0,1]$, and

$$f_{n+1}(x) := \begin{cases} \dfrac{1}{2} f_n(3x), & \text{if } x \in [0, 1/3], \\ \dfrac{1}{2}, & \text{if } x \in [1/3, 2/3], \\ \dfrac{1}{2} + \dfrac{1}{2} f_n(3x-2), & \text{if } x \in [2/3, 1]. \end{cases} \tag{1}$$

Note that the graphic of f_1 is a polygonal through the 4 points $(0,0)$, $(1/3, 1/2)$, $(2/3, 1/2)$, $(2/3, 1)$. Similarly the graph of f_2 is a polygonal through the 8 points

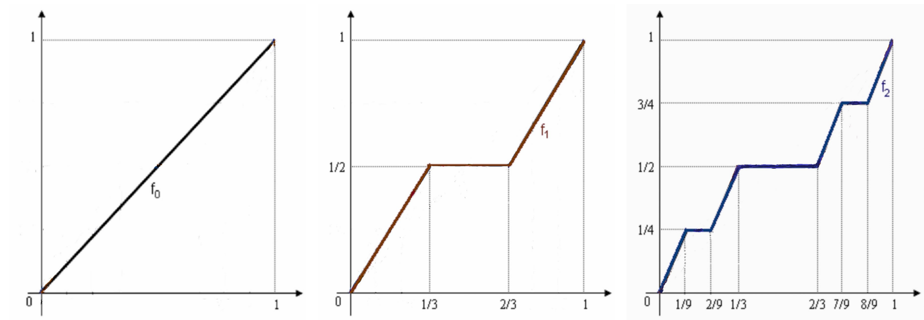
$$(0,0),\ \left(\frac{1}{9}, \frac{1}{4}\right),\ \left(\frac{2}{9}, \frac{1}{4}\right),\ \left(\frac{1}{3}, \frac{1}{2}\right),\ \left(\frac{2}{3}, \frac{1}{2}\right),\ \left(\frac{7}{9}, \frac{3}{4}\right),\ \left(\frac{8}{9}, \frac{3}{4}\right),\ (1,1),$$

as Fig. 1(c) shows. It is then possible to determine the graphic of f_n as a polygonal through 2^{n+1} points, giving them in a recursive way as for formula (1).

We can also say that the graph of f_1 is made up by 2 oblique line segments equal to the diagonal of the rectangle $\left[0, \frac{1}{3}\right] \times \left[0, \frac{1}{2}\right]$ except for translations parallel to the axes, and 1 horizontal segment line. Then, the graph of f_2 is constituted by 2^2 oblique segment lines equal to the diagonal of $\left[0, \frac{1}{3^2}\right] \times \left[0, \frac{1}{2^2}\right]$ except for translations parallel to the axes, and $1 + 2 = 2^2 - 1$ horizontal line segments. In general we can write the following

Remark 1. For all $n \in \mathbb{N}_0$, the graph of f_n is made up by 2^n oblique line segments equal to the diagonal of the rectangle

$$\left[0, \frac{1}{3^n}\right] \times \left[0, \frac{1}{2^n}\right]$$



(a) $f_0(x)$. (b) $f_1(x)$. (c) $f_2(x)$.

Fig. 1. The first three approximations of the Cantor-Vitali function.

except for translations parallel to the axes, and $2^n - 1$ horizontal line segments.[4] Hence, the graph of f_n consists of a total of $2^{n+1} - 1$ line segments through 2^n points, whose ends are $(0, 1)$ and $(1, 1)$.

For the area subtended by the functions f_n the result is trivial. If we define

$$a_n := \int_0^1 f_n(x)\, dx$$

for all $n \in \mathbb{N}_0$, we have

$$a_0 = \frac{1}{2} \qquad \text{and} \qquad a_{n+1} = \frac{1}{3} + 2 \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot a_n \quad \forall n \in \mathbb{N}_0, \tag{2}$$

from which we get

$$a_n = \frac{1}{2} \quad \forall n \in \mathbb{N}_0.$$

Note that at the same result for a_n we can arrive through considerations of symmetry of f_n with respect to the point $(1/2, 1/2)$.

Now let l_n be the length of the graph of the function f_n, where $n \in \mathbb{N}_0$. From Fig. 1(a-c) it is immediate that

$$l_0 = \sqrt{2}, \qquad l_1 = \frac{1 + \sqrt{13}}{3}, \qquad l_2 = \frac{5 + \sqrt{97}}{9}. \tag{3}$$

To find a formula for l_n we cannot use the same method seen in (2) for a_n. This because the area of a figure changes linearly with respect to both its width (say x-size) and its height (say y-size), but the length of a curve does not. Obviously, the length of a curve does not vary linearly even with respect to the distance

[4] We can obviously specify the number of segments length 1/3, 1/9, etc., but this is not relevant in this paper and it is not relevant for the subsequent computation of l_n.

between its two extremes. A simple example is obtained by comparing the two ratios

$$\frac{l_1}{\|(1,1)\|} = \frac{1+\sqrt{13}}{3\sqrt{2}} \approx 1.085539$$

and

$$\frac{\lambda}{\left\|\left(\frac{1}{3},\frac{1}{2}\right)\right\|} = \frac{\frac{1}{9}+2\sqrt{\left(\frac{1}{9}\right)^2+\left(\frac{1}{4}\right)^2}}{\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{1}{2}\right)^2}} = \frac{2+\sqrt{97}}{3\sqrt{13}} \approx 1.095427,$$

where $\|(x,y)\| = \sqrt{x^2+y^2}$ is the Euclidean norm of a vector $(x,y) \in \mathbb{R}^2$ and λ denotes the length of the graph of f_2 moving from $(0,0)$ to $(1/3,1/2)$.

To find a formula for l_n, therefore, we use another strategy. Recalling Remark 1, we know that the graph of f_n is made up by 2^n oblique line segments, i.e. 2^n diagonals of length

$$\sqrt{\left(\frac{1}{3^n}\right)^2+\left(\frac{1}{2^n}\right)^2}, \tag{4}$$

and 2^n-1 horizontal line segments for a total length equal to

$$1-2^n\cdot\frac{1}{3^n} \tag{5}$$

(just remove from the length of the unit segment $[0,1]$ the projections on the x-axis of the 2^n oblique segments). Therefore, from (4) and (5) we conclude that

$$l_n = 1-\left(\frac{2}{3}\right)^n+\sqrt{1+\left(\frac{2}{3}\right)^{2n}}, \tag{6}$$

for all integers $n \geq 0$. In fact, we note that, for $n = 0,1,2$, we recover respectively the values in (3).

3 Highlights using the grossone system

Using standard analysis, all we can say on the behavior of the sequence (6) when n approaches infinity is that

$$\lim_{n\to\infty} l_n = \lim_{n\to\infty} 1-\left(\frac{2}{3}\right)^n+\sqrt{1+\left(\frac{2}{3}\right)^{2n}} = 2, \tag{7}$$

and this is consistent with the observation that taking x- and y-projections of the $2^{n+1}-1$ line segments which constitute the graph of f_n, they go to cover the two segments

$$[0,1]\times\{0\} \qquad \text{and} \qquad \{0\}\times[0,1]$$

without overlays, and the slope of the oblique segments approaches $+\infty$.

In the remainder of this section, as already announced in the Introduction, we will use the grossone-based system introduced by Sergeyev in the early 2000s to say something different and more precise than the limit in (7).

The grossone-based numeral system, roughly speaking, is founded on two different fundamental units: the ordinary unit 1 which give rise to natural numbers, integers, rationals, etc. and a correspondent "infinite unit" ① called *grossone*, which generate a whole range of infinite numbers like

$$①, \quad 2①, \quad -①, \quad -5①, \quad \frac{3}{7}①, \quad -\frac{9}{7}①, \tag{8}$$

and also numbers with a finite and an infinite part like

$$① + 6, \quad 3① - 1, \quad -① + \frac{4}{3}, \quad -5①^2 + 7① - \sqrt{6}, \quad \frac{8}{7}①^4 - \frac{5}{7}① + 63, \tag{9}$$

etc. Introducing ① implies to introduce also infinitesimal numbers, i.e. inverses of infinite numbers, like

$$\frac{1}{①}, \quad ①^{-3/2}, \quad 5①^{-6} + 7①^{-5/2}, \quad ①^{-5/2} - \frac{① + 2}{①^3 - 4① + 4}, \tag{10}$$

and, obviously, additions, multiplications, divisions between elements as in (9) and (10).

Much more details on the grossone-based system can be found in Sergeyev's papers [30, 33] or the popular book [29]. In recent years many applications of the grossone system have been found, for example to fractals (see [4, 7, 8, 32]) and *blinking fractals* (see [12, 31]), summations, ordering, probability, game theory (see [13–16, 27, 34] and the references therein), optimization, differential equations, Infinity Computer (see [1, 5, 17, 19]), and many other fields. Links with logic, mathematics foundations, Fibonacci numbers and unimaginable ones can be found in [9–11, 21–23]), and very interesting are also a series of recent didactic studies and experimentations in schools about the grossone systems (see [2, 3, 18, 20, 24–26, 28]).

Applying grossone to Cantor-Vitali function allows us to make mare precise computations. For the areas a_n we have no changes because it is constant for all n. Instead, for the length l_n of the graphic of the approximation function $f_n(x)$, we get a different result depending on the infinite level n. For instance, if $n = ①$ we get from (6)

$$l_① = 1 - \left(\frac{2}{3}\right)^{①} + \sqrt{1 + \left(\frac{2}{3}\right)^{2①}}, \tag{11}$$

which is strictly less than 2, but it differs from it by an infinitesimal quantity. We can also decrease this infinitesimal difference by considering chained sequences (see [29, 30, 33]). For example, in virtue of the monotonicity of (6), clear from the geometric discussion made above, we get that

$$l_{5①/2+6} = 1 - \left(\frac{2}{3}\right)^{5①/2+6} + \sqrt{1 + \left(\frac{2}{3}\right)^{5①+12}} \tag{12}$$

is strictly greater than (11) and strictly less than 2. Now it should be clear how it is possible to give precise numerical values not only to infinite quantities, but also to infinitesimal ones. Making comparisons between them is then straightforward through the new system.

Acknowledgments

This research was partially supported by Seeweb s.r.l., Cloud Computing provider based in Frosinone, Italy, and part of DHH Group.

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