Unimaginable numbers and Infinity Computing at school: an experimentation in northern Italy

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Abstract. In this paper we describe an experimentation carried out in a high school in northern Italy. The focus is to present to students new concepts of very big numbers and the infinite, studying their response, approaches, intuit and the suitability to apply in larger scale. In particular the new concepts and notations regard unimaginable numbers and the infinity computing. Several exercises have been suggested to students arousing much interest. Also a final test has been proposed and it is discussed in part in this paper. Among many observations and conclusions, we confirm a great ease of use of infinity computing by students and an almost immediate and intuitive degree of reception. Also unimaginable numbers, hyperoperations and Knuth's powers proved highly educational value, but they are more difficult to master.

Keywords: Mathematical education · Infinity computing · Unimaginable numbers · Student learning

1 Introduction

In the literature, it is possible to find several projects that concern the development of new learning approaches in the context of high schools. These have different purposes, such as promoting greater motivation for learning, developing skills and propensities for critical and creative thinking.

In this paper we describe an experimentation carried out in 2023 in a high school in Treviso, northern Italy. This study was conducted in parallel and simultaneously with a similar one carried out in a second high school in Crotone, southern Italy (see [19] in this same volume for details). The aim of this study in two schools was to investigate the students' response to some teaching activities concerning new approaches to very big numbers and infinity. In particular we employed infinity computing similarly as in 2019 in three Italian schools (see [2] and [18]), but now, for the first time to our knowledge, employing in a mathematical education research also the so-called *unimaginable numbers* which are almost infinite numbers (see Sect. 2 for some details and references).

Specifically, this paper is structured as follows. Sect. 2 gives, in a quick way, some basic material, information and references on unimaginable numbers and the arithmetic of infinity obtained by adopting the *grossone*-based system. Sect. 3 is the core of the paper and describes the experiment conducted in 2023 at the *IIS "Palladio"* in Treviso, North Italy. It had as its object the study of the didactic approach of high school students in the face of unimaginable numbers and the arithmetic of infinity.

2 Mathematical tools and references

A natural number is called *unimaginable* if it is greater than 1 *googol*, i.e. 10^{100}. The ordinary exponential or scientific notation is far to be able to write unimaginable numbers, so, in the XX Century many special notational methods were developed. Among them we recall *Knuth's up-arrow notation, hyperoperation notation, Conway chained arrow notation, Moser-Steinhaus notation*, etc. For details, examples, basic definitions, etc., we refer the reader to [6, 12, 20] and the references therein.

The name *infinity computing* is referred to a new computational system proposed by Y. Sergeyev about 20 years ago and able to perform computations with infinite and infinitesimal numbers very easily and handily. This is a great strength of Sergeyev's new system, which allows an immediate approach even towards high school students. It is commonly called the *grossone-based computational (or numerical) system* because it is constructed on the fundamental unit ①, precisely called *grossone*, as well as on the ordinary unit 1. The former allows to write infinite and infinitesimal numbers in the same way as 1 allows to write finite ordinary numbers (see [26, 27, 29] for introductory surveys on the new system). There are many applications of Sergeyev's new paradigm in several fields of mathematics, physics and applied sciences. For instance [1, 4, 29] contain applications to differential equations and optimization, [3, 7, 8, 13, 14, 28] applications to summations and fractals, [15, 16] employ the new system combined with cellular automata, [9, 21, 29, 32, 33] contain investigations on its mathematical foundations and some discussions about new views of classical paradoxes, applications to logic, etc. In [11] there are also some hints to apply infinity computing to the Carboncettus sequence which originates from Fibonacci numbers (see [10]).

Furthermore, in the previous edition of this conference, NUMTA 2019, a special session of new computational tools and math education has begun to catch on (see for example the paper [30, 31] and others), and inside it also some researches on the possible employ of the grossone system in high schools (see [2, 18]). More recently the book [25] and the papers [17, 22–24] have been published, and [19] in this same volume "continues" the article [18] but involving also unimaginable numbers. In the next subsection the reader can find more details on them.

3 Activities and tests in Treviso

3.1 Description of the experimentation

The involved classes in the experimentations at the IIS "Palladio" in Treviso were two fourth classes, one with 25 students (10 male and 15 female) and another with 22 (14 male and 8 female). The total number of students is 47 (24 male and 23 female), and the age is between 17 and 18 y.o. This research, together with the twin one in Crotone [19], can be considered a second step of [2], which had been organized in 2019 on the basis of [5].

In particular, we proposed to both classes a short cycle of lessons, 7 or 8 hours divided in 4-6 days. Just about 1 hour was about the grossone-based system, and the remaining ones on unimaginable numbers (this means that the grossone system can be used almost at an intuitive level, see the conclusions in Subsect. 3.2 and cf. [2, 18]). During the lectures on unimaginable numbers, tetrations, pentations, etc., many examples and exercises had been proposed to students. A final test with several questions was also administered a week after the conclusion of the cycle of lessons. Each question had 3-5 predefined multi-choice answers, only one of them correct. A selection of 8 questions with the number and percentages of answers by the students are reported in Tables 1-8 below. For the reader's convenience the (unique) correct answer will be specified each time.

Using progressive numbering, and not the original one, for the 8 questions that we report as examples, Question 1 asked to find the correct claim about ① + 1 among three different possibilities shown in the first column of Table 1. In the second and fourth column of Table 1, M and F denote the number of male or female students, respectively, who have opted for the corresponding choice. In columns three and five we find the percentage relative to the column M and F, respectively. Column six, denoted by "Tot.", reports the total number of students who opted for this choice (i.e., "Tot.= M+F", roughly speaking), and column seven, the last one in Table 1, reports the percentage relative to the column "Tot.".

Question 2 (for us) asks about the result or meaning of ① + ①, giving three possible choices shown in the first column of Table 2. The subsequent columns 2-7 play the same role as in Table 1.

Table 1. Question 1, about ① + 1, had three possible suggested answers: (b) is the correct one.

Possible choices	M	M %	F	F %	Tot.	%
(a) It is not a number	2	8.33	5	21.74	7	14.89
(b) It is greater than any natural number	17	70.83	17	73.91	34	72.34
(c) It is equal to ①	5	20.83	1	4.35	6	12.77

Table 2. Question 2 about ① + ①: (a) is the correct answer.

Possible choices	M	M %	F	F %	Tot.	%
(a) It is equal to 2①	11	45.83	15	65.22	26	55.32
(b) It is equal to ①	12	50	5	21.74	17	36.17
(c) It is not a number	1	4.17	3	14.04	4	8.51

Table 3. Question 3 about ①/2: (b) is the correct answer.

Possible choices	M	M %	F	F %	Tot.	%
(a) It is equal to ①	2	8.33	5	21.74	7	14.89
(b) It is a positive infinite number	18	75	18	78.26	36	76.60
(c) It is equal to $10^{2\,000\,000\,000\,000}$	4	16.67	0	0	4	8.51

Question 3 asks about the writing ①/2 giving three possible choices as in the first column of Table 3. Question 4 instead asked to find the correct claim among four possible choices as in the first column of Table 4.

Question 5 asked to find the correct claim among four possible choices listed in the first column of Table 5. Question 6 asked for the correct value of $2 \uparrow\uparrow 2$ among four possible choices listed in column 1 of Table 6. Then Question 7 asked to find the correct value of the pentation $2 \uparrow\uparrow\uparrow 2$ among five possible options listed in the first column of Table 7. Finally, Question 8 asks about two Knuth's powers, $3 \uparrow 3$ and $2 \uparrow\uparrow 2$, giving five different options as in column 1 of Table 8. Columns 2-7 in Tables 3-8 play obviously the same role as in Table 1.

3.2 Data analysis and conclusions

A first interesting and quite singular result emerges from Table 1: among male and female students, we find something like "an X configuration" in the answers. In fact, note that the correct answer (b) was given by an equal number of male and female students (i.e. 17), while the numbers relating to answers (a) and (c) are exactly exchanged in the form of X if we consider the integers 1 and 2 the closest integer approximations of the average percentage value 6.34%.[4] This simple observation could be the starting point for a series of more in-depth researches on the differences between male and female students in their conception, vision, intuition, previous experiences, stimuli and interest in the study of infinity in mathematics in the broadest sense of the term.

For Question 2 we expected many more answers (a). Although the majority of students, and especially female students, answered correctly, answer (b) still proved to be attractive. The percentage of correct answers to Question 3 rose to 76.60% (from 55.32% of Question 2): *a priori* we would have expected the

[4] In other words, consider the three average percentage values taken in the shape of an X: we find 72.39% (average value between 70.83% and 73.91%), 21.28 (average value between 20.83% and 21.74%), 6.34% (average value between 8.33% and 4.35%). Calculating the closest integer approximations on the basis of 24 male and 23 female students, we obtain precisely the values that appear in columns M and F of Table 1.

Table 4. Question 4: (d) is the correct choice.

Possible choices	M	M %	F	F %	Tot.	%
(a) ① is less than $+\infty$	14	58.33	15	65.22	29	61.70
(b) ① is greater than $+\infty$	2	8.33	4	17.39	6	12.77
(c) ① is equal to $+\infty$	4	16.67	1	4.35	5	10.64
(d) ① and $+\infty$ are not comparable	4	16.67	3	13.04	7	14.89

Table 5. Question 5: (b) is the correct option.

Possible choices	M	M %	F	F %	Tot.	%
(a) A googol is greater than ①	2	8.33	5	21.74	7	14.89
(b) A googol is less than ①	15	62.5	16	69.57	31	65.96
(c) A googol is equal to ①	3	12.5	1	4.35	4	8.51
(d) Googol and grossone are not comparable	4	16.67	1	4.35	5	10.64

opposite! It would be very interesting to conduct future studies with this pair of questions and with larger samples of students.

In Question 4, the non-comparable answer probably requires more reflection, and is objectively more difficult and less intuitive. In Question 5, the percentage of correct answers is excellent, close to 70% for girls. It is interesting to note how the answer (d) on non-comparability had almost the same percentages of choices as for Question 4. Indeed, exactly 4 boys chose (d) for Question 4 and for Question 5: are they the same? Are they less reflexive than the girls who have gone from 3 to 1 in parallel? On points like this there would be much to study and debate; an extra fact that we report is that male students seem less inclined to do lengthy calculations and reflections, and were on average quicker in completing the test.

In Question 6 only about 14.9% of students answered correctly, and the percentage of male students is higher in this case. The preference of option (c) is not so clear, it is probably a trick question for them. Turning instead to the pentation $2\uparrow\uparrow\uparrow 2$ considered in Question 7, it seems really difficult to explain the answer (a) chosen by 8 boys (and 0 girls). Maybe something like the so-called *anchoring bias* for (less thoughtful and faster) male students? It should be noted that almost all of the girls chose the complicated expression appearing in (d) and it is a very interesting phenomenon. "If (d) is that complicated, then it must be

Table 6. Question 6 about $2\uparrow\uparrow 2$: (b) is the correct option.

Possible choices	M	M %	F	F %	Tot.	%
(a) It is equal to 2^4	4	16.67	3	13.04	7	14.89
(b) It is equal to $2 \cdot 2$	4	16.67	3	13.04	7	14.89
(c) It is equal to 2^{2^2}	14	58.33	16	69.57	30	63.83
(d) None of the above	2	8.33	1	4.35	3	6.38

Table 7. Question 7 about the pentation $2\uparrow\uparrow\uparrow 2$: (e) is the correct answer.

Possible choices	M	M %	F	F %	Tot.	%
(a) It is equal to 0	8	33.33	0	0	8	17.02
(b) It is equal to $2^{2\,000\,000\,000\,000\,000\,002}$	0	0	0	0	0	0
(c) It is equal to 2002	0	0	0	0	0	0
(d) It is equal to $2^{(2^2)^{2^{(2^2)^{2^{(2^2)}}}}}$	13	54.17	21	91.30	34	72.34
(e) None of the above	3	12.5	2	8.70	5	10.64

Table 8. Question 8 on two Knuth's powers, $3\uparrow 3$ and $2\uparrow\uparrow 2$. Obviously (c) is here the correct choice.

Possible choices	M	M %	F	F %	Tot.	%
(a) $3\uparrow 3$ is less than $2\uparrow\uparrow 2$	10	41.67	16	69.57	26	55.32
(b) $3\uparrow 3$ and $2\uparrow\uparrow 2$ are not comparable	0	0	0	0	0	0
(c) $3\uparrow 3$ is greater than or equal to $2\uparrow\uparrow 2$	14	58.33	7	30.43	21	44.68
(d) $3\uparrow 3$ and $2\uparrow\uparrow 2$ are equal	0	0	0	0	0	0
(e) None of the above	0	0	0	0	0	0

the right one...", or what is technically called the *seductive detail bias* could be at play.

The answers given to Question 8 show that Knuth's arrow notation, even only for tetrations, needs a medium-long assimilation time by the students. For Question 8 the percentage of correct answers by male students is much higher (about twice).

We conclude the paper with some final remarks and considerations. During all the class-activity, also after the body of the experimentation, the students have shown great facility in the correct use of the gross-based system (cf. also [2] and [18]). It must be taken into account that almost all the time of the lessons before the test was devoted to unimaginable numbers, tetrations, etc., which require, as confirmed by the test, much more time to be understood or used in simple contexts. Nonetheless we are convinced of the high educational value they can provide as a sort of "mind gym" at various levels.

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