# Some notes on a continuous class of octagons

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**Abstract.** This paper deals with a continuous class of octagons, built from the so-called "sequence of Carboncettus octagons". We first explain the building techniques, then we study the main properties and make comparisons with the original sequence (which gives a discrete family of octagons) and some related ones. Finally, in the last part of this work, we also study the behavior of this new continuous family of octagons by adopting the lens of the infinity computing.

**Keywords:** Octagons · Fibonacci numbers · Golden section · Carboncettus octagons · Infinity computing.

## 1 Introduction

This paper contains some notes on a new continuous family of octagons that arises from the so-called sequence of *Carboncettus octagons* (see [10, 11]). But while the sequence of Carboncettus octagons is a discrete family, in this paper we will study a family that originates from it, and whose elements depend on a parameter that varies in a continuous interval, i.e. an interval of real numbers.

We call our continuous family and its element *Carboncettus-like octagons*, often abbreviated in the following as *CL octagons*. For the historical origins of the name *Carboncettus* we refer the reader to [29].

In this work we also make some comparisons between the new octagons family, the original sequence $\{C_n\}_n$ of Carboncettus octagons, and the sequence of the "normalized" Carboncettus octagons $\{C_n^N\}_n$ (see [11]). In this perspective the use of infinity computing assumes a very important role and gives very interesting advantages, as we show in Sect. 4.

In order to construct our sequences, for some calculations and for the figures necessary to our discussion, we have used the software *GeoGebra* (see [18, 19]). We point out that, in this context, both the use of infinity computing and the use of a software like GeoGebra can have very interesting educational implications. For example, they could be very useful in the study of the transition phase from discrete to continuum mathematics (see [17, 20, 23, 24, 34–37]).

## 2 The original octagons sequences $\{C_n\}_n$ and $\{C_n^N\}_n$

In this section we recall the construction of the original sequence $\{C_n\}_n$ of Carboncettus octagons (see [11, 10]). The building procedure is as follows:

- We take two concentric circles with their centers at the origin of the axes and we draw four tangent lines to the inner circumference, each one parallel to a coordinate axis.
- The points where the tangents intersect the outer circle are the vertices of an octagon.

The main characteristic of the sequence $\{C_n\}_n$ is that each octagon is obtained by using two consecutive Fibonacci numbers, but both with odd or even indexes. In other words, the $n$-th Carboncettus octagon $C_n$ is originated by starting from an inner circumference of radius equal to the $n$-th Fibonacci number $\varphi_n$, and an outer circumference of radius $\varphi_{n+2}$. We remark that at least starting from $C_4$, all the octagons of the sequence $\{C_n\}_n$ are almost regular, i.e. completely indistinguishable from a regular octagon (see [11, 10, 29]).

Regarding to the sequence $\{C_n^N\}_n$, the normalized radii of the circumferences are $\varphi_n/\varphi_n$ (for the internal radius) and $\varphi_{n+2}/\varphi_n$ (for the external one). For the external normalized radius we get:

$$\lim_{n\to\infty} \frac{\varphi_{n+2}}{\varphi_n} = \lim_{n\to\infty} \frac{\varphi_{n+2}}{\varphi_{n+1}} \cdot \frac{\varphi_{n+1}}{\varphi_n} = \varphi^2,$$

where

$$\varphi = \frac{1+\sqrt{5}}{2} \tag{1}$$

is the golden ratio and, consequently,

$$\varphi^2 = \frac{3+\sqrt{5}}{2}. \tag{2}$$

Hence we can conclude that the sequence $\{C_n^N\}_n$ converges to a "limit octagon" inscribed inside a circumference with radius $\varphi^2$.

## 3 The new continuous family of octagons

In this section we present a new class of octagons, whose constructive model is derived from the one used for $\{C_n\}_n$ and $\{C_n^N\}_n$. Then we will discuss some properties and characteristics of the elements of the new family. In Fig. 1 the building model for the CL octagons is represented. In Fig. 2 a detail of the CL octagon when $r = 0.1$.

### 3.1 General characteristics of the CL octagons

We can summarize the building procedure for the new class of octagons in the following steps:

- We identify two concentric circumferences.
- We draw the two horizontal tangents and the two vertical tangents to the inner circumferences.

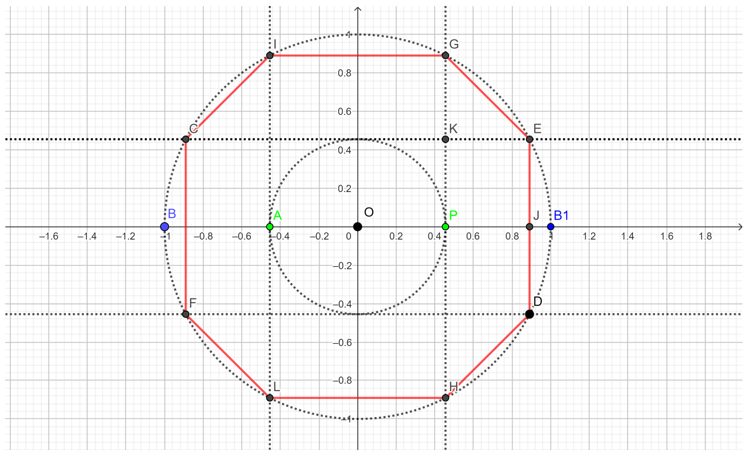


**Fig. 1.** The output of GeoGebra for the CL octagons construction.

- Each tangent line intersects the external circumference at two points.
- We join these points and find an octagon.

The resulting octagon is called a *CL octagon* to remember the used procedure to build it. For this new class of octagons the outer circle is kept fixed with unit radius, while the radius of the inner circle varies *continuously* within the interval $[0, 1]$.

At an operational level, with reference to Fig. 1, we can say:

- It arises by construction

$$|OB| = |OB_1| = 1, \qquad |OP| = r, \qquad r \in [0, 1].$$

- Always by construction it arises

$$|AP| = 2r.$$

It can be observed that the construction conditions of the *CL* octagons are absolutely compatible with geometric constraints of general validity. For example, the chord $|OB|$ will always be not greater than $|OB_1|$, where the last is the radius of the outer circumference.

Since

$$|OB| = |OB_1| = 1,$$

a consequence of Pythagoras theorem is

$$|OJ| = \sqrt{1 - r^2}. \tag{3}$$

Then we find

$$|PJ| = |OJ| - |OP| = \sqrt{1 - r^2} - r. \tag{4}$$

From the properties of parallelograms we get

$$|GK| = |KE| = |PJ| \tag{5}$$

and then

$$|GE| = \sqrt{2}|PJ| = \sqrt{2}(\sqrt{1-r^2} \ - r). \tag{6}$$

So it is possible to identify three functions, defined below, which *measure* in some way the ratios between the sides of the octagon CL thus obtained:

$$\theta(r) = \frac{|ED|}{|GE|} = \frac{2r}{\sqrt{2}(\sqrt{1-r^2} \ - r)} = \frac{\sqrt{2}r(\sqrt{1-r^2}+r)}{1-2r^2}, \tag{7}$$

$$\rho(r) = \frac{|GE|}{|ED|} = \theta(r)^{-1} = \frac{(\sqrt{1-r^2}-r)}{\sqrt{2}r}, \tag{8}$$

$$\delta(r) = |GE| - |ED| = \sqrt{2}(\sqrt{1-r^2}-r) - 2r. \tag{9}$$

In this way it is possible to study the evolution of these CL octagons when the radius $r = |OP|$ of the inner circumference varies. In the next section we will studies the properties and the behavior of these functions.

### 3.2 The functions $\theta(r)$, $\rho(r)$ and $\delta(r)$

Some important information can be deduced from the study of the functions $\theta(r)$, $\rho(r)$ and $\delta(r)$ defined above. First of all it should be noted that the functions will be studied in relation to their "geometric meaning". For $\theta(r)$, (see (7)) we have:

$$\begin{cases} 1-r^2 \geq 0 \\ 1-2r^2 \neq 0. \end{cases} \tag{10}$$

The function $\rho(r)$ instead represents the inverse ratio, in comparison with $\theta(r)$. Referring to $\rho(r)$ (see (8)), we have:

$$\begin{cases} 1-r^2 \geq 0 \\ r \neq 0. \end{cases} \tag{11}$$

Regarding to $\delta(r)$ (see (9)) we get:

$$1-r^2 \geq 0. \tag{12}$$

Hence we can rewrite the functions $\theta(r)$, $\rho(r)$ and $\delta(r)$ as follows

$$\theta: \ D_\theta(r) \to \mathbb{R}, \qquad r \mapsto \frac{|ED|}{|GE|},$$

$$\rho: \ D_\rho(r) \to \mathbb{R}, \qquad r \mapsto \frac{|GE|}{|DE|},$$

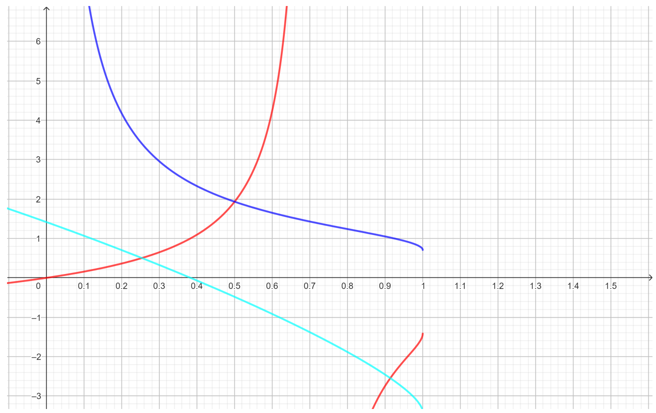


**Fig. 2.** The graph of the function
$\theta(r)$ (red), $\rho(r)$ (blue) and $\delta(r)$ (light blue).

$$\delta: \ D_{\delta(r)} \to \mathbb{R}, \qquad r \mapsto (|GE| - |DE|),$$

where, by solving (10), (11) and (12), we get

$$D_{\theta(r)} = [0,1] \setminus \left\{ \frac{\sqrt{2}}{2} \right\}, \tag{13}$$

$$D_{\rho(r)} = (0,1], \tag{14}$$

$$D_{\delta(r)} = [0,1]. \tag{15}$$

We now make some considerations for the functions $\theta(r)$ and $\rho(r)$. Note that the function $\rho(r)$ does not exists when $r = 0$, and the same for the function $\theta(r)$ when $r = \sqrt{2}/2$. Geometrically this occurs with the disappearance of one of the sides $|DE|$ or $|GE|$, respectively. When the internal circumference has radius $r = 0$, it becomes a single point and the CL octagon becomes a square. Note that when $r = \sqrt{2}/2$ the CL octagon becomes a square as well.

By convenience, we denote an element of the new continuous family of octagons by $O_r$, where the subscript $r \in [0,1]$ represents the radius of the internal circumference. In Fig. 2, the graphs of the functions $\theta(r)$, $\delta(r)$ and $\rho(r)$ are represented by different colors.

Obviously all the elements of the sequence $\{C_n^N\}_n$ is contained in the new continuous family $\mathcal{O} := \{O_r : r \in [0,1]\}$. In other words, this means that every normalized Carboncettus octagon $C_n^N$, for all natural numbers $n$, can be retrieved in the new family $\{O_r : r \in [0,1]\}$ for a suitable value of the parameter/radius $r$. In fact, to get the $n$-th normalized Carboncettus octagon $C_n^N$ we need the value $r = \varphi_n/\varphi_{n+2}$ for the continuous parameter $r$. A particular case occurs when $\delta(r) = 0$ (see (9)), which yields

$$r = \frac{\sqrt{2 - \sqrt{2}}}{2}. \tag{16}$$

| $n$ | $r_{int}(C_n)$ | $r_{ext}(C_n)$ | $r_{int}(C_n^N)$ | $r_{ext}(C_n^N)$ | $\varphi_n/\varphi_{n+2}$ |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 1 | 2 | 1/2 |
| 2 | 1 | 3 | 1 | 3 | 1/3 |
| 3 | 2 | 5 | 1 | 5/2 | 2/5 |
| 4 | 3 | 8 | 1 | 8/3 | 3/8 |
| 5 | 5 | 13 | 1 | 13/5 | 5/13 |
| 6 | ... | ... | ... | ... | ... |
| $n \to \infty$ | ... | ... | ... | $\varphi^2$ | $1/\varphi^2$ |

**Table 1.** In the second column $r_{int}(C_n)$, which is equal to $\varphi_n$, represents the value of the internal circumference used to construct the n-th Carboncettus octagon $C_n$. Similarly, $r_{ext}(C_n)$ is the radius of the external one. Them, the meaning of $r_{int}(C_n^N)$ and $r_{ext}(C_n^N)$ are now obvious.

A further important fact is that we can find the "limit normalized octagon" $C_\infty^N$ in our family $\mathcal{O} = \{O_r : r \in [0,1]\}$, see Eq. (19) below.

In order to understand better the evolution of the octagon $O_r$ when the parameter $r$ varies, we divide the unitary segment $[0,1]$ into the following 5 different subsets:

$$[0,1] = \left[0, \frac{1}{3}\right) \cup \left[\frac{1}{3}, \frac{1}{2}\right] \cup \left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right) \cup \left\{\frac{\sqrt{2}}{2}\right\} \cup \left(\frac{\sqrt{2}}{2}, 1\right]. \tag{17}$$

Note that dome special octagons can now easily recovered in the suitable subset appearing in the decomposition (17). For example, all the octagons of the sequence $\{C_n^N\}_n$ lie in the interval

$$\left[\frac{1}{3}, \frac{1}{2}\right], \tag{18}$$

and this is a quite remarkable fact. In particular, note that the minimum of the interval (18) gives $C_2^N$ and the maximum $C_1^N$, i.e. the first two elements of the sequence $\{C_n^N\}_n$. Within the interval (18) we also find the limit octagon $C_\infty^N$ obtained in correspondence of the value

$$r = \frac{1}{\varphi^2}, \tag{19}$$

(see (2)). Note also the singleton $\{\sqrt{2}/2\}$ in the decomposition (17): we in fact already know that the value $r = \sqrt{2}/2$ corresponding to the case where $O_r$ degenerate to a square. The same phenomenon and the same square (with side $\sqrt{2}/2$) occurs for $r = 0$ too, but the square has its vertices on the coordinate axes in this case. Fig. 3 shows the CL octagon $O_{1/10}$ which is close to coinciding with the square $O_0$.

Recall that in (16) we found the value of $r$ correspondent to the regular octagon, i.e.

$$r = \frac{\sqrt{2-\sqrt{2}}}{2} \approx 0.38268343.$$

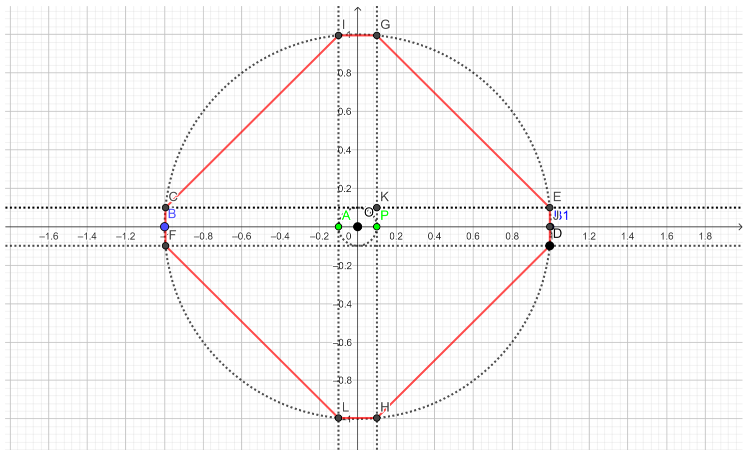


**Fig. 3.** The CL octagon $O_{1/10}$ obtained by choosing $r = 1/10$.

Note that even such a value of $r$ belongs to the "principal interval" $[1/3, 1/2]$. Finally we remark another important fact: the last of the five subsets shown in the decomposition (17), i.e. the interval $(\sqrt{2}/2, 1]$ corresponds to the most strange octagons, where they lose their convexity (see Fig. 4). Looking at Fig. 4, this phenomenon is due to the fact that a line segment like $DE$ will become secant for the inner circumference when $r > \sqrt{2}/2$. In fact, in the range of values $(\sqrt{2}/2, 1]$ for $r$ we find a family of *self intersecting octagons*.

To resume ideas, in Table 2 we list some of the most representatives octagons $O_r$ belonging to our family $\mathcal{O}$.

## 4 The sequence $\{C_n\}_n$ $\{C_n^N\}_n$ viewed through the lens of infinity computing

In this section we want to give some hints for the study of the original Carboncettus sequence $\{C_n\}_n$ when $n$ grows, by using a newly introduced methodology called *infinity computing* or *grossone*-based numerical system.

In the early 2000s Y. Sergeyev introduced a new numerical system able to perform computations with infinite and infinitesimal number, in a very easy and handle way, as we ordinarily do with natural and real numbers. Roughly speaking, such a new system is constructed on two fundamental units: the ordinary 1 for finite numbers and a new unit ①, called *grossone*, for infinite (and infinitesimal) quantities. We refer the reader to [31, 33] for introductory surveys and also to [30] for a book written in a popular way. The grossone-based system has today a number of applications in pure and applied mathematics, as well as in experimental sciences. For example see [1, 5, 15, 16, 33] for applications to differential equations, game theory and optimization, [4, 7, 8, 13, 14, 28, 32] to fractals, space filling curves and summations, [9, 26, 33] for some discussions on foundations

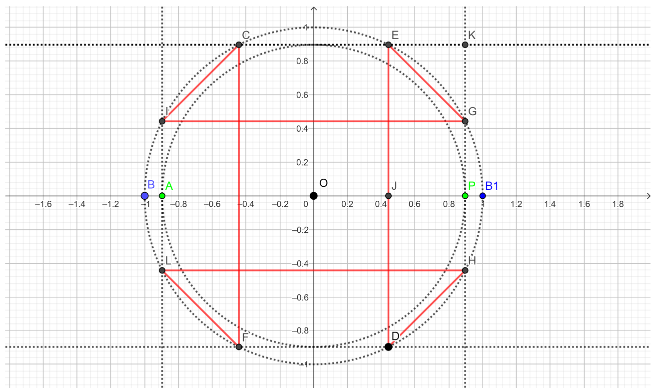


**Fig. 4.** The CL octagon $O_{9/10}$ obtained by choosing $r = 9/10$. It is a self-intersecting polygon.

and paradoxes, etc. In the last few years Sergeyev's system has also been used in didactic experiments in high schools (see for example [2, 3, 21, 22]) and a very interesting connection with Fibonacci numbers has been explored in [27]. Regarding to [2, 22], innovative educational experimentation joining *unimaginable numbers* and *grossone* are described in [12, 25]. As said in the previous sections, in [11] there are some hints to apply infinite computing and grossone to the sequence $\{C_n\}_{n\in\mathbb{N}}$. Following them, we first point out to the reader that Fibonacci sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ has a least element $\varphi_{①}$ in the grossone system, and it can be written for instance as

$$\varphi_{①} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{①} - \left(\frac{1-\sqrt{5}}{2}\right)^{①}}{\sqrt{5}}. \tag{20}$$

The previous formula is derived from the well known Binet formula for $\varphi_n$ (see [6]). The very relevant thing is that, through the new system, we can precisely compute various measures of infinitely large octagons. For example the last element $C_{①}$ of the $\{C_n\}$ octagons sequence has the following diameter

$$\mathrm{diam}(C_{①}) = 2\varphi_{①+2} = \frac{\left(1+\sqrt{5}\right)^{①+2} - \left(1-\sqrt{5}\right)^{①+2}}{2^{①+1}\sqrt{5}}, \tag{21}$$

which, for instance, is easily comparable with the one of $C_{①-2}$:

$$\mathrm{diam}(C_{①-2}) = 2\varphi_{①} = \frac{\left(1+\sqrt{5}\right)^{①} - \left(1-\sqrt{5}\right)^{①}}{2^{①-1}\sqrt{5}}. \tag{22}$$

The reader can note that, without the new system, we can just say that

$$\lim_{n\to\infty} \mathrm{diam}(C_n) = +\infty. \tag{23}$$

| Value of $r$ | Element of $\{C_n^N\}_n$ | Resulting figure |
|---|---|---|
| 0 | NA | Square |
| 1/3 | $C_2^N$ | Octagon |
| 3/8 | $C_4^N$ | Octagon |
| ... | ... | Octagon |
| $\varphi^2 = \frac{3+\sqrt{5}}{2}$ | $C_\infty^N$ | Limit octagon |
| $\sqrt{\frac{2-\sqrt{2}}{2}}$ | NA | Regular octagon |
| ... | ... | Octagon |
| 5/13 | $C_5^N$ | Octagon |
| 2/5 | $C_3^N$ | Octagon |
| 1/2 | $C_1^N$ | Octagon |
| 1 | NA | Square |

**Table 2.** Values of the radius $r$ of the internal circumference associated with $O_r$ (first column ) and the relative value of $n$ as an element of the sequence $\{C_n\}_n$, when such $n$ exists (second column).

Many more computations and evaluations are allowed by the new grossone-based system and with different levels of precision. For example, if we denote by $\approx_i$ the equality up to infinitesimals, we have from (21) and (22)

$$\operatorname{diam}(C_{①}) \approx_i \frac{\left(1+\sqrt{5}\right)^{①+2}}{2^{①+1}\sqrt{5}}, \qquad \operatorname{diam}(C_{①-2}) \approx_i \frac{\left(1+\sqrt{5}\right)^{①}}{2^{①-1}\sqrt{5}}.$$

And we can compute quite easily, up to infinitesimals, the perimeter, the area, and other measures of $C_{①}$, $C_{①-2}$, etc. Anyway, a deeper discussion of such things and more demanding calculations are beyond the scope of this paper: we plan to do this and to give full examples in a future work.

## 5 Conclusions

In this work we introduced a new family of octagons $\mathcal{O}$ whose elements vary continuously on dependence of a real parameter $r \in [0, 1]$. We showed that the previous known Carboncettus normalized octagons $C_n^N$ introduced in [11], are all recoverable $\mathcal{O}$ for suitable values of $r$ belonging to the subinterval $[1/3, 1/2] \subset [0, 1]$. We called an element of the greater family $\mathcal{O}$ a Carboncettus-like octagon (CL octagon for short). We have studied the main properties of the new family $\mathcal{O}$, also with the help of three suitably defined functions $\theta(r)$, $\delta(r)$ and $\rho(r)$, but much work remains to do and we aim to do so in the near future.

In Sect. 4 we finally used the grossone based system to perform a first study of the original sequence $\{C_n\}_n$ inside the greater family $\mathcal{O}$ when $n$ goes to assume infinite values. In this direction there are many aspects that can be investigated in the future as well.

Even possible employs, of the material here exposed, in mathematical education can be examined in the future. And, lastly, from a theoretical and purely mathematical point of view, possible connections with Blaschke's theorem and subsequent results could be studied.

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