Unimaginable numbers: a Case Study as a starting point for an educational experimentation

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Abstract. How big is a large number? How far can our imagination go to think or to imagine a large number? This work is part of the study of didactic approaches aimed at the knowledge of unimaginable numbers in secondary school. The topic is very interesting because this numbers very large, but finite numbers can be transitioned in the passage of the concrete transition from the concept of finite to that of infinity. This work is placed in continuity with the previous case studies carried out on the computational arithmetic of infinity. Specifically, a further case study is shown here: the planning phase contains identification regarding research questions and additional aspects including choice of case study method and discovering strengths or limitations. The case study design linked together from the beginning until the end, including everything from hypothesis and question design to data analysis and conclusion. In addition, it is showed the unit of analysis concerning the case study, the method by which these numbers together with their characteristics will be exhibiting order to give an idea of their estimated size, even with regard to the physical universe. Finally, the effects and results in terms of educational implications will be evaluated through the analysis of the results that emerged from the administration of a questionnaire proposed to the students of the classes involved.

Keywords: Unimaginable numbers · Knuth up-arrow notation · grossone

1 Introduction

The regions of Southern Italy known as Magna Graecia have made a great contribution from a scientific-philosophical point of view to the study of infinity. Evidently already at the time the most illustrious thinkers were used to asking themselves questions such as the following:

- How big is a large number and ?
- How far can our imagination go to think or imagine a large number?

The answers to these questions allow us to introduce the topic of this work: the so called *unimaginable numbers*. From a historical point of view, the idea of manipulating extremely large numbers, what today we could somehow define as the forerunners or ancestors of unimaginable numbers, has deep roots. We can say that the first of those numbers that today we would call unimaginable was a quantity proposed by Archimedes of Syracuse. This number is well known as Ψ. It could also be found under the name of *Arenarius* or *Send Reckoner*. We have to note that the right name is Ψ while different names are due to a misunderstanding between the name of this number and the name of the book in which the same number is described. We well call this number Arenarius'Ψ (or simply Ψ) from now on. Regarding to this number, we find $\Psi = 10^{8\times 10^{16}}$ (from the homonyms work, as we have already said, that is simply a number represented by *1* followed by *80 billions millions zeros*, a number used to identify the grain contained in the entire universe). To date, the best known notation for their representation is the one called "Knuth's up-arrow notation". More than a known notation in an absolute sense, it is a relatively known notation since this sector of mathematics is still little traveled in the academic field and almost unexplored in the rigorously scholastic field. This work is aimed to present an unknown topic as unimaginable numbers, to secondary school students and evaluate the didactic results by analyzing results emerged fro a a test carried out by the same students. We want to remember that we don't submit a *zero knowledge* test. Students carried out the test after a presentation of unimaginable numbers and their properties. Such an approach has already been proposed for other topics. In particular, possible forms of experimentation related to the computational arithmetic of infinity have been carried out (see [3, 18]). The aim of this work is therefore similar. The object of the study is no longer a form of computational arithmetic of infinity (①) but a form of arithmetic that stops an instant before: the arithmetic of unimaginable numbers. For these reasons we divide this work into three principal sections. Each of them is a fundamental part of the case of study and needs of a detailed description. In Section 2 we present unimaginable numbers and their properties. we have chosen to make a detailed presentation, starting from a meticulous description of the historical part and of both the purely algebraic and the analytical parts (including the hyperoperations, the different representations); besides we represents some properties as the results of exercises proposed to the students. We re proposed these example even on this paper, for discursive completeness, without leaving any step that could be useful to repeat a similar work in future. In Section 3 we do a little presentation of the properties of computational arithmetic of infinity (①), and then, finally, in Section 4 we recognize the results of the test, after a presentation of the set of questions which constitute the same test.

2 Unimaginable numbers

In this section we describe unimaginable numbers and their properties in the same way they have been presented to the students of the sample before they carried out the test. The unimaginable numbers are finite but extremely large quantities. In this section, the unimaginable numbers will be presented.

2.1 Historical and scientific background

Since the period of the Hellenic colonization of southern Italy, some schools that arose in the colonies dealt with representing extremely large numbers. Subsequently, it seems that all traces referring to this topic have been lost in official and sector literature. In fact we return to talk about unimaginable numbers only from 1900 thanks to the work of some of Hilbert's disciples (W. Ackermann) and then G. Sudan, R. Robinson, R. Peter and finally by R. Goodstein around 1940. Then again a blank period until the works of D. E. Knuth (1976). In this section, some ways of representing unimaginable numbers will be described. For some of these modes of representation some arithmetic properties are also foreseen as amply demonstrated (see [11, 12, 21, 22]); only a mention will be given to these latter properties given that the objective of the work is to proceed with a case of study preparatory to a real didactic experimentation aimed at disseminating the topic in the school environment. The basic idea is to express a number, large or small, as the result of an operation suitably coded in such a way as to be defined *recursively*. We introduce the concept of *hyperoperation*, already originally proposed by Ackermann. It is developed starting from the Ackermann function. In a such way, any operation can be expressed recursively starting from known results. About Ackermann function, it is as follows:

$$f_A : \mathbb{N}^2 \to \mathbb{N}, \qquad (m;n) \mapsto f_A(m;n). \tag{1}$$

It is defined recursively as follows:

$$\begin{cases} f_A(0;n) & = 1; \\ f_A(m+1;0) & = f_A(m;1); \\ f_A(m+1;n+1) & = f_A(m;f_A(m+1;n)). \end{cases} \tag{2}$$

Starting from Ackermann function we can define hyperoperations as follows:

$$H_n(a;b) = \begin{cases} b+1 & \text{if} \quad n=0; \\ a & \text{if} \quad n=1 \quad \text{and} \quad b=0; \\ 0 & \text{if} \quad n=2 \quad \text{and} \quad b=0; \\ 1 & \text{if} \quad n\geq 3 \quad \text{and} \quad b=0; \\ H_{n-1}(a;H_n(a;b-1)) & \text{otherwise.} \end{cases} \tag{3}$$

We have to remember that $H_n(a;b) = a[n]b$. By previous definition for hyperoperation we can reach important results (whose demonstration has been gift to students as an exercise!):

$$H_1(a;1) = a+1. \tag{4}$$

$$H_n(a;1) = a \qquad \forall n \geq 2. \tag{5}$$

Knuth came to propose a simple method for expressing sufficiently or arbitrarily large quantities as the result of well-known and well-defined operations. From Ackermann notation to Knuth's up-arrow notation:

$$\uparrow^n = \underbrace{\uparrow\uparrow ... \quad ... \uparrow}_{n \quad \text{copies}} = H_{n+2} \tag{6}$$

since

$$n_{Ackerman} = n_{Knuth} + 2. \tag{7}$$

According to (6) and (7) we will intend

$$a \uparrow^n b = a \underbrace{\uparrow\uparrow ... \quad ... \uparrow}_{n \quad \text{copies}} b = H_{n+2}(a;b). \tag{8}$$

A such representation is known in the reference literature as *Knuth's up-arrow notation*:

$$a \uparrow^n b = \begin{cases} a \times b \qquad \text{if} \quad n = 0; \\ 1 \qquad \text{if} \quad n \geq 1 \quad \text{and} \quad b = 0; \\ a \uparrow^{n-1} (a \uparrow^n (b-1)) \quad \text{if} \quad n \geq 1 \quad \text{and} \quad b \geq 1. \end{cases} \tag{9}$$

Such a definition provides that operations are defined recursively. We conclude that the general term of (9) is

$$a \uparrow^n b = a \uparrow^{n-1} (a \uparrow^n (b-1)). \tag{10}$$

Stating from (6) and (7) students will simply proof (as an exercise!) *Knuth's version* for (4) and (5). Another very important exercise is to prove that:

$$a \uparrow^n b = \underbrace{a \uparrow^{n-1} (a \uparrow^{n-1} (\ldots (a \uparrow^{n-1} a) \ldots))}_{b-1 \text{ copies of } a\uparrow^{n-1} \text{ and one of } a}. \tag{11}$$

Note that the the innermost operation (inside $(b-2)^{th}$ brackets) is $(a \uparrow^{n-1} a)$.
Observation
This mode of representation, known as Knuth's vertical arrows representation is also known as *krata*. The term krata is the plural form of *kratos*, an ancient word of Greek origin which means *power* ([7, 12]). We have to highlight that kratos is a function *and it is only for simplicity that we use it as Knuth's vertical arrows notation.* Therefore, as far as the representation in the form of power is concerned, the hyperoperation can therefore be expressed in the form $B \uparrow^d T$ for which:

- B is the *base*
- d is the *depth* i.e. which has to be understood as a "power"

- T is the *tag* and indicates the number of copies of the operation defined not by d but starting by d

In detail, the base represents the numerical value on which we perform the operations, the depth is somehow which is associated with the operations themselves and finally the tag gives a measure of the iterations. We must pay attention to this last aspect as the tag does not really constitute the measure of the iterations of the operations, that is the number of repetitions of the operations themselves if not in a particular representation. In fact, there are at least three equivalent representations in krata or Knuth's up-arrow notation. These three different equivalent representations can be generalized by (11). For discursive completeness, some practical results will now be presented which are useful for giving a measure of the capacity of compact representation or, if you like, of the computational power of the notation that makes use of Knuth's arrows.

Product $(n = 0)$

$$a \uparrow^0 b = a \times b$$

or *sum* between a and $b - 1$ *copies* of a or, definitively, *sum* of b *copies* of a.

Exponentiation $(n = 1)$

$$a \uparrow^1 b = \underbrace{a \times a \times ...a}_{b \quad \text{copies}} = a^b$$

or *product* between a and $b-1$ *copies* of a or, definitively, *product* of b *copies* of a.

Tetraction $(n = 2)$

$$a \uparrow^2 b = H_4(a; b) = \underbrace{a^{a^{\cdots a}}}_{b \quad \text{copies}}$$

or *exponentiation (power)* between a and $b - 1$ *copies* of a or, definitively, *power (recursive!)* of b *copies* of a.

Pentation $(n = 3)$

$$a \uparrow^3 b = H_5(a; b) = \underbrace{a \uparrow^2 (a \uparrow ...(a \uparrow^2 a))}_{b \quad \text{copies}}$$

or *tetraction* between a and $b - 1$ *copies* of a or, definitively, *tetraction (recursive!)* of b *copies* of a.

Exaction $(n = 4)$

$$a \uparrow^4 b = H_6(a; b) = \underbrace{a \uparrow^3 (a \uparrow ...(a \uparrow^3 a))}_{b \quad \text{copies}}$$

or *pentaction* between a and $b-1$ *copies* of a or, definitively, *pentaction (recursive!)* of b *copies* of a.
Of course the process can be iterated over and over again.

For example

$$3 \uparrow^3 5 = 3 \uparrow^2 (3 \uparrow^2 (3 \uparrow^2 (3 \uparrow^2 3))).$$

Since $3 \uparrow^2 3 = 3^{27} = 7625597484987$, we can conclude that

$$3 \uparrow^3 5 = 3 \uparrow^2 (3 \uparrow^2 (3 \uparrow^2 7625597484987))$$

2.2 Some famous very big numbers

In this section we will present some large and more or less known numbers. From a purely historical point of view it can be said that the progenitor of the family of unimaginable numbers is Archimedes' Arenarius'Ψ.

Arenarius Ψ
It is equal to $\Psi = 10^{8\times 10^{16}}$. In Knuth's notation: $\Psi = ((10 \uparrow 8) \uparrow^2 2) \uparrow (10 \uparrow 8)$. This number is so big that it is represented, in decimal notation, with 80 *million billion* digits (first digit is 1 and the others are 0). A word page, written in *Calibri* font and with size 11 has about 4000 characters (exactly 3956). We need

$$\frac{8\times 10^{16}}{4\times 10^{3}} = 2\times 10^{13}$$

sheets, for a height of $2\times 10^{13}\times 8\times 10^{-5} = 1.6\times 10^{9} m$ (more than 4 times the distance between Hearth and Moon!).
Googol and *googolplex*.
The googol number is a quantity introduced by the American mathematician Edward Kasner in 1938 to give an estimate of the unimaginable magnitude of infinity in comparison with large but finite quantities. *Googol* is equal to 10^{100} and it represents an upper bound for the size of the physical universe (since the number of elementary particles of the physical universe does not go beyond 10^{90}). This means that a googol is about 10 *billion* times the number of particles of physical universe. Starting from googol we define googolplex as follows: $10^{googol} = 10^{10^{100}}$.
Mega and *megiston*.
The megiston (⑩ according to Steinhaus-Moser representation) is a very large unimaginable number. It is smaller than Graham's number. The Steinhaus-Moser representation for mega is ②.
Graham's number (G).
Graham's number, the greatest unimaginable numbers which has been used for a mathematical demonstration. Among the various unimaginable numbers, it is the one that presents itself as a solution to a problem which is Graham's problem (hence the name Graham's number!). In Knuth's up-arrow notation, the

Graham number is defined by the following recursive representation:

$$G = \left.\begin{matrix} 3\underbrace{\uparrow \dots \uparrow}3 \\ \vdots \\ \underbrace{} \\ 3 \uparrow^4 3 \end{matrix}\right\} 64 \text{ levels,} \tag{12}$$

where the number of arrows appearing at each level from the second onwards is given by the number expressed in the next lower level. In other words we have $G = g_{64}$ where g_n is recursively defined by

$$\begin{cases} g_1 = 3 \uparrow^4 3 \\ g_n = 3 \uparrow^{g_{n-1}} 3 \quad \text{if } n \geq 2 \end{cases} .$$

In order to give an idea of the size of the Graham number, we must remember that $g_1 = 3 \uparrow^4 3$ represents an *exaction* that is $g_1 = 3 \uparrow^4 3 = 3 \uparrow^3 (3 \uparrow^3 3)$. In next section we will speak about other properties of unimaginable numbers.

2.3 When a number could be consider *unimaginable*?

In this section, we will present some arithmetic properties of unimaginable numbers and a *threshold* for unimaginable numbers.

- When an obviously large number can be considered unimaginable?

This question is important as it allows us to have an effective estimate, beyond the numerical data itself but related to a more profound interpretation of it, of the kratic representation. Therefore, if we set the *googol* as the minimum limit, as the *threshold*, we find that there are only 58 ([7, 12]) numbers that have a *non-trivial* kratic representation (i.e. in the form of non-trivial powers according to Knuth's notation) and which at the same time are lower than the set limit. Conversely, if the threshold rises further and sets the limit 10^{10000} then it has been demonstrated ([12]) that there are only 2893 numbers below the threshold 10^{10000} that have a non-trivial kratic representation. Of these, 2888 are of the type $a \uparrow^2 2$ while the remaining 5 do not have a representation of this type. Still for unimaginable numbers, given the function $k(BDT) = B \uparrow^D T$, i.e. the function that associates a kratic representation to each triad of the type *(Base,Depth,Tag)*, various arithmetic properties exist and have been tested, in particular with respect to a periodicity with respect to modular arithmetic. For example, if we suppose $B, D, T \geq 2$, the sequences:

- $\{B \uparrow^D n\}_n$
- $\{B \uparrow^n T\}_n$
- $\{B \uparrow^n n\}_n$

they become constants modulo M (for a fixed positive integer M, see [8]). Also, the sequences $\{n \uparrow^D T\}_n$; $\{n \uparrow^D n\}_n$; $\{n \uparrow^n T\}_n$ and $\{n \uparrow^n n\}_n$ are all periodic modulo M. Finally, there is an algorithm to be able to calculate $\{B \uparrow^D T\}$ modulo M (see [8]). Finally, there are alternative representations with respect to the kratic representation. Among these we can mention the *box notation*, the *superscript and subscript notation* (see [10, 11]), the *extended operations* ([12]), *Nambiar notations* ([13]), or *Cutler's bar notation* also called *Cutler's circular notation* (see [14, 15]). At the end we find *Conway's chained arrows*: which could be used similarly to Knuth's up-arrow notation or when a number is to big that even Knuth's notation could be inappropriate!

3 The *grossone*-based numerical system

About 20 years ago, Y. Sergeyev introduced a new computational system based on the so-called *grossone*, whose symbol is ①. This new system is able to perform computations not only using the ordinary (finite) real number, but also by using infinite and infinitesimal quantities. Sergeyev's system is also very easy to use as the familiar system of natural or real numbers. Roughly speaking, the grossone-based system is made up on two fundamental units: the familiar unit 1 to obtain finite numbers (integers, rationals and reals) and a new unit ①, called *grossone*, which is used to write infinite and infinitesimal numbers. The reader can find many details on the new system in introductory surveys as [29, 31] and also in the book [27] written in a popular way. Since the new system was proposed about 20 years ago, it has immediately found a large number of applications in many fields of both mathematics and experimental sciences. For example see [1, 5, 15, 16, 20, 31], for connection with Fibonacci numbers and applications ([24, 19]), for applications to ordinary differential equations, optimization, cellular automata and game theory, [4, 8, 9, 13, 14, 26, 28, 30] for applications to fractals, space filling curves and summations, [10, 23, 31] for some discussions on logic foundations, paradoxes and their solutions, etc. Recently the grossone-based system has been also tested for educational purposes in high schools both in Italy and abroad: see for example [2, 3, 17, 25] and [18] where the same school as this paper was involved). In the next section and in the conclusions we will discuss and compare the results obtained 4 years later in the same school, Liceo Scientifico Statale Filolao (KR, Italy), on similar tests regarding very basic computations involving the *grossone* system.

4 The *case of study*

In this section we present the core of the case of study i.e. the results of tests in order to measure the effectiveness of a didactic approach (discussed in Section 2) related to the knowledge of unimaginable numbers. After discussing of the topics of the case of study we presents the sample of the students, the test and its results. The basic idea is to present the unimaginable numbers to high school students. The didactic proposal starts from two elements: the history of unimaginable numbers and the need for a compact representation that overcomes the computational constraint of decimal representation but also of exponential representation. The students are then introduced to the concept of hyperoperations, i.e. operations defined recursively with references to operations already known and sufficiently used (for example, zeroing instead of the successor, *unaction* instead of addition and so on). For completeness, the known unimaginable numbers will also be presented with examples capable of giving an idea of their magnitude. Finally, in order to evaluate the didactic impact, the sample will be administered a test and then the results will be presented and discussed. We have to remember that for the aim of this work we used the model proposed in [6] which has been adapted and for the needs of a high school.

4.1 The sample

The sample object of the case study discussed in this work is made up of the pupils of 2 classes of the Filolao scientific high school of Crotone. Pupils are at fourth class. In details we are submitted the test to the students of two classes for a total of 48 pupils.

4.2 The test

The test to be administered to the pupils who make up the sample of the case study is a test of 10 questions with multiple choice. The questions tend to evaluate the following indicators:

- the acquisition of skills in the representation of unimaginable numbers;
- the knowledge of some famous unimaginable numbers and therefore the knowledge of the aspects that could arouse more curiosity or that could act as catalysts towards the knowledge of these particular aspects of mathematics;
- sensitivity towards the measurement of unimaginable numbers arithmetic skills in the operations of transition from one form of representation to another.

The test is administered anonymously in the form of multiple choice questions. Both the questions and the associated answers will be distributed randomly in order to avoid cheating phenomena. The set was elaborated in the same way as a set of questions proposed for the investigation of the arithmetic of infinity (*grossone*). Since in [18] it has been demonstrated that it is useful to administer the questionnaire after the cycle of lessons, we decided to carry out the test only after the presentation of the arguments. Now we present a sample of questions and multiple choice answers referred both to *grossone* test and to to *unimaginable numbers* test.
Grossone test (an example of)

1 Let consider the expression $3① + 4①$. It's equal to (select among multiple choice):

A 7①.
B 4①.
C ①.
D without sense.
E neither of the previous answers.

5 Let consider the expression $A = ①$, $B = ① + ①$ e $C = ① \times ①$. Select the right order relationship:
A $A < B < C$.
B $A < C < B$.
C $B < C < A$.
D $B < A < C$.
E $A = B = C$.

9 The number of elements of $\mathbb{Z}$ set is (select among multiple choice):
A ①.
B $① - 1$.
C $① + 1$.
D ∞.
E neither of the previous answers.

The questions number 1, 5, 9 are the original ones in the students' test.

Unimaginable numbers test (an example of)

1 *megiston* is
A The biggest knonw unimaginable number as solution of a known problem.
B The number whose value is $10^{8\times10^{16}}$.
C ⑩.
D ②.
E ①.

3 What does the following notation represent

$$\begin{cases} g_1 = 3 \uparrow^4 3 \\ g_{64} = 3 \uparrow^{63} 3 \end{cases} \tag{13}$$

A Arenarius' Ψ.
B googlplex.
C mega.
D megiston.
E Graham's number.

4 The result of $H_1(a; b)$ is:
A 0.
B a.
C $a \times b$.

D $a + b$.
E $b + 1$.

5 The result of $H_3(a; b)$ is
A a.
B a^b.
C $a \times b$.
D $a + b$.
E $b + 1$.

4.3 The results

In this section we presents test results. After a short presentation of unimaginable numbers and their properties pupils were invited to carry out the test on that topics and a second test referred to *grossone*. Both the tests have been presented in previous sections. We did not submit a *zero knowledge* test. In such a way we could have a direct measurement of the effectiveness of the presentation of the topic (since unimaginable numbers represent a topic which is not covered in high schools). In figures n. 1 and 2 we highlight the results of single students regarding to both the tests. Since the size of the sample is small, 48 units or students (for this reason we present a case of study) we can presents single results for each students. This method give us an instantaneous idea of the didactic impact on a single students. In fact, the scores achieved by individual pupils have led to averages specific to the spheres of excellence. We can therefore conclude the following: a detailed theoretical introduction accompanied by some significant examples and some elements of the history of mathematics, used to capture the students' attention, provided the starting point for reaching acceptable levels of knowledge. This result, certainly positive, must certainly be contextualized to the single case study but it can certainly be taken as a model for new and more in-depth experiences in the didactic field. We remember that every test related to this work is made up by ten questions(while for [18] we have 46 questions); for every question, every students scores 1 point for each correct answer and zero points for each non given or incorrect answer. We remember *grossone* test the expected value is 8.67/10 points while for *unimaginable numbers* test it is 7.92/10 points. For [18] the expected value is 42.59/46 points. We have to note that for *grossone* test we find a mean value which reaches 87 percent of maximum value (normalized on a set of 10 questions) while in [18] it reaches 92 percent but it is related to a greater set of questions (46 questions instead of 10). For *unimaginable numbers* test we have an expected value of 79 percent (strictly closed to *grossone* test results). Besides, in the better case we have to remember that the same test has been carried out twice (before speaking about *grossone* and after!) and on a greater sample (3 classes instead of 2). These are very important results since if we consider, over and over again, that nobody of the students in the sample have any knowledge about unimaginable numbers before this didactic experimentation.

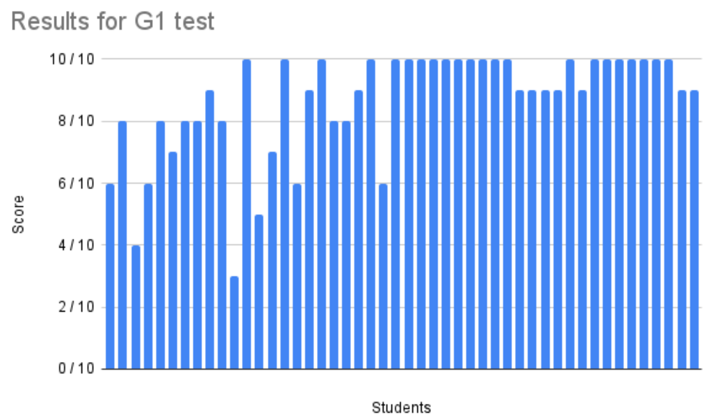

Fig. 1. Statistics and results for *grossone* test

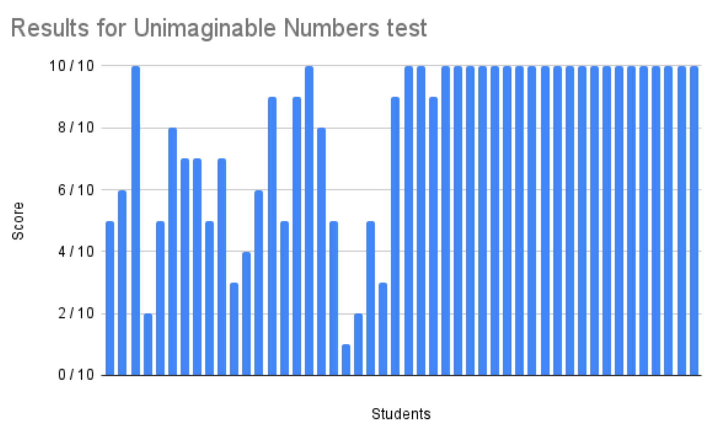

Fig. 2. Statistics and results for *unimaginable numbers* test

5 Conclusions

In this work we presents the results of a case of study carried out on students of Italian high school. We expose and some elements of theory related to unimaginable numbers (first of all the idea of hype operations and notations used to define them). We have to remember a very important aspects: unimaginable numbers are not studied or known by students of Italian high school. Nevertheless the test administered had a very positive outcome. The results (the *positive results* as shown in previous section!) of this test and the approach of this case of study, similarly to the previous ([18]) should be used in order to do other case of study for unexplored topics in secondary school or to be a starting points for didactic experiment involving a great number of students. We have to remember that this work is only a case of study and a starting point for future educational aims. For this reason we can use it for further and future step:

- *zero knowledge* and *not zero knowledge* tests only related to unimaginable numbers
- *zero knowledge* and *not zero knowledge* tests related to both unimaginable numbers and *grossone*
- comparison between previous tests results

We remember that this is only a starting point from which we can lead different didactic experimentations by using well defined proceedings.

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