Legal systems and fractals, towards infinity computing

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Abstract. The focus of this work is to take an approaching step, or rather, try to create a stronger connection than those existing between jurisprudence and mathematics. In particular, between legal systems in the broadest generality and fractal structures, using in particular the von Koch curve and infinity computing which allows a precise measurement of infinite quantities, and therefore of stretches of fractal curves with infinite length.

Keywords: Legal systems · Fractals · Hausdorff dimension · Fractal curves · Von Koch curve · Grossone.

1 Introduction

In the Modern Era we have witnessed the consolidation of an orientation towards distinct disciplines of study and different fields of research with well-defined boundaries, in contrast to the ancient view that sought the unity of knowledge. Today, we are beginning to see a new reversal of this trend, aimed at recovering a community of views and an exchange of tools between the technical-scientific and humanistic disciplines. This article aims to make a small contribution to this process by proposing a rapprochement between two fields of knowledge that are generally considered to be distant: law and mathematics. Specifically, it aims to propose an unusual conference between one of the most recent topics in mathematics, namely fractals and the fractal geometry developed by them, and the intimate essence of the legal reasoning and argumentation. The first to apply a geometric view to jurisprudence was J. Balkin in 1986 in an article entitled *"The Crystal Structure of Legal Thought"* [6]. The author speaks of "the crystalline structure" because in 1986 he was not yet aware of the fractals and the results that Mandelbrot and others had achieved in the previous two decades, as he himself specifies in a subsequent article of over 120 pages in 1991 *"The promise of legal semiotics"* [7]. In fact, in 1967, B. Mandelbrot, with the publication of the article *"How long is the coast of Britain?"* [20] had opened up a new way of interpreting many physical objects and structures found in nature. Many other papers and books followed [20] in the next two or three decades, for example the very famous book [21] by Mandelbrot himself or [15, 18], but a real explosion in the research and applications of fractals has occurred in the last quarter of a century.

As regards fractals and legal systems, in 2000, two eminent researchers, D. Post and M. Eisen, the first a legal scholar, the second a computational biologist, published an article clearly inspired by [20] entitled *"How long is the coastline of the law? Thoughts on the fractal nature of legal systems"* [25]. More recently another single article [34] was published in 2013 by A. Stumpff on the relations between fractals and legal systems. To our knowledge, no other works different from the quoted ones exist until [19] in 2022 by one of the author of this paper.

The purpose of this work is to take it one step further into investigating the possibility of study legal systems with metric and geometric tools. We continue the ideas proposed in [6, 7, 25, 34] and [19] into the direction of using fractals to interpret the internal structure of jurisprudence and legal systems. In particular we use the von Koch curve as fractal model to represent and, in some sense, encode internal paths joining different points inside legal systems, bodies of law, legal argumentation, and jurisprudence in general. A further powerful tool that we can use to push on the ideas of introducing concrete and numerical metrics into legal systems is the infinity computing introduced by Sergeyev. His new methodology allows to assign precise infinite numerical values, written with the help of a new infinite numerical unit called *grossone*, to generic infinite quantities and sizes. This is just what we need to numerically evaluate the distance of two points on the von Koch curve, where by distance we mean the length of the path obtained by walking along the curve and not the Euclidean distance in the plane. We conclude that, by measuring sections of fractal curves we can find somewhat a metric inside legal systems: using the expression of Post and Eisen, we can precisely measure all pieces of "the coastline of the law" in the same way we can give a precise infinite numerical value to a piece of real coastline or to the perimeter of an island (if we are able to mathematically model its shape).

Furthermore, the capacity to introduce a metric into legal systems allows to study them through the use of fractal or Hausdorff dimension as proposed in [19].

As regards the structure of this paper, Sect. 2 gives to the reader some basic notations and examples how to write infinite numbers through the grossone-based system and some basic references, with particular attention to fractal applications. In Sect. 3 we recall the construction of the von Koch curve and use it to explain and represents paths inside legal systems.

In this paper we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{R}$ to denote the set of positive, non-negative integers and real numbers, respectively.

2 Infinity computing and fractals

In the early 2000 Y. Sergeyev introduced a new numerical system which allows computations not only with ordinary finite numbers (e.g., natural, rationals, reals, etc.) but also with infinite and infinitesimal ones. Sergeyev's system is built on two fundamental units: the ordinary 1 to generate finite numbers, and a new infinite unit ① called *grossone* which generates infinite numbers and infinitesimal ones (through its reciprocal $①^{-1} = 1/①$). Examples of infinite and infinitesimal numbers in the new system are the following

$$2①, \quad 5①/9, \quad 4①^2 - 5①, \quad -6①/11 + 8①^{1/2}, \quad 6①^{-2}, \quad -4①^{-1} + 5①^{-3}.$$

Consider now the following number

$$a = -\frac{2}{3}①^3 + 5①^{2/3} - 7 + 3\sqrt{2} - 6①^{-1/2} + \frac{5}{4}①^{-2};$$

it has an infinite part $(-\frac{2}{3}①^3+5①^{2/3})$, a finite one $(-7+3\sqrt{2})$ and an infinitesimal part $(-6①^{-1/2}+\frac{5}{4}①^{-2})$. To perform computations using the common operations is very easy and intuitive in the grossone system. This is also proved by several researches which propose Sergeyev's system in some Italian high schools, using even zero knowledge tests with very satisfactory results: see for instance [3, 16] and [2] in this volume. See also the very recent papers [22, 23] and the book [26]. The reader can find extensive details on the new system into introductory surveys by Sergeyev himself as [27, 32] or [29] also in Italian.

The new system, or methodology, has been successfully applied to a number of areas both of mathematics, physics, biology and applied sciences. For example, [1, 5, 10, 13, 14, 17, 32, 33] apply the grossone-based systems to ordinary differential equations and optimization, cellular automata, game theory, and logic paradoxes. But for the arguments discussed in this paper, the mist important applications are those relative to fractals and fractal curves (see [4, 8, 9, 11, 12, 24, 28, 30]), and in particular to von Koch curve (see [31]) which occupies a central role for our discussion, see Sect. 3.

3 The von Koch curve and legal constructions

3.1 Construction of the von Koch curve

Let K_0 be the unitary interval $[0,1] \times \{0\}$ contained in $\mathbb{R}^2$. We obtain K_1 by dividing K_0 into three parts and by replacing the central one (i.e. $[1/3, 2/3] \times \{0\}$) by the two sides of the equilateral triangle with base $[1/3, 2/3] \times \{0\}$ and a vertex in $(1/2, \sqrt{3}/6)$. In other words K_1 is the polygonal shown in Fig. 1(a) whose vertices, in order from left, are

$$(0,0), \quad \left(\frac{1}{3}, 0\right), \quad \left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right), \quad \left(\frac{2}{3}, 0\right), \quad (1,0).$$

To obtain K_2 we apply the same procedure used to get K_1 from K_0 to each of the four line segments constituting the polygonal K_1: the result is shown in Fig. 1(b).

(a) K_1. (b) K_2. (c) K_3.

Fig. 1. The first three steps starting from K_0 in the construction of von Koch curve K.

Iterating this process n times we obtain a polygonal K_n composed of 4^n line segments all with the same length equal to $(1/3)^n$. Therefore we can deduce the following formula for the length $l(K_n)$ of the polygonal K_n,

$$l(K_n) \;=\; \left(\frac{4}{3}\right)^n \tag{1}$$

for all non-negative integers n. For instance, K_3 is a polygonal line composed by $4^3 = 64$ line segments of length $(1/3)^3 = 1/27$, as shown in Fig. 1(c). The only thing you have to pay attention to is that of constructing the equilateral triangles on the correct part every time, that is, an observer who travels the segment K_0 from $(0, 0)$ to $(1, 0)$, he sees the curve K_1 to his left in the middle third $[1/3, 2/3] \times \{0\} \subset \mathbb{R}^2$. And the same when an observer travels along the curve K_n from $(0, 0)$ to $(1, 0)$ for all $n \in \mathbb{N}_0$: he sees the curve K_{n+1} to his left where K_n does not coincide with K_{n+1}.

Using the Hausdorff distance as usual in fractal geometry, the sequence $\{K_n\}_{n\in\mathbb{N}_0}$ of closed compact sets contained in $\mathbb{R}^2$ is proved to be a Cauchy sequence, hence it will converge to a set K called the *von Koch curve*. Hence K is the limit

$$K \;:=\; \lim_{n\in\mathbb{N}} K_n.$$

3.2 Paths inside legal systems

Legal arguments, the typical constructs of legal disciplines, legal systems and legal corpus, certainly do not follow a straight line to go from one point to another, however one understands them and whatever the two points in question represent. Similar observations have been made in the past and from different points of view in [6, 7, 25, 34] and [19], as we mentioned in the Introduction.

We shall now try to examine, with an example, the tortuous ramifications typical of legal arguments. By way of example, we shall consider the case of a dispute concerning the exploitation of a patent for an invention and try to show how legal arguments can branch off, in this situation, *ad infinitum*, without ever potentially reaching an end point or an end (see [19]). This will result in the approximation of this logical-spatial construct to the fractal geometry and in particular to the von Koch curve, or rather to its approximations K_n.

Let us suppose that plaintiff A accuses defendant B of improperly using a certain patent or reproducing its contents without having the rights to do so.

At a first level, the level of maximum generality, one may ask whether or not B is liable for infringing ah intellectual property or copyright covered by a patent. But immediately thereafter, at the second level the question branches off or deviates from an ideal straight line like successive approximations K_n of the von Koch fractal curve.

Consider, for example, four branches or four changes of direction such as the following:

1. Is applicant A really the owner of the copyright or the invention?
2. Is the object protected by the patent or the patented process really subject to a regular patent?
3. Are the proceeds derived from that invention or innovation really subject to a patent or patentability?
4. Has B infringed one or more rules against A?

Defendant B could at this point argue along one, two, three or all four of the branches exemplified above, producing other strands and other jagged line. For example, B could divide issue 1 into three sub-issue as follows:

1.1. Is the invention or innovative process in question truly original? Or is it a development of something already existing or known?
1.2. In the patented product fully and incontrovertibly result of A's ingenuity or work?
1.3. Do the relevant laws permit the product in question to be protected or covered by a patent?
1.4. Is the patent temporally valid and in existence at the time or during the alleged infringements?
1.5. Have the rights in question been transferred?

The list could continue with further branches or deviation of "level two" and, in addition, each of the points in the list could give rise to further branches or deviations of "level three".

For example, item 1.3 could give rise the following branches:

1.3.1. Does the patent in question fully comply with the statutory indications and requirements?
1.3.2. Does what A would like to be protected coincide with what is covered by the patent?
1.3.3. Does the benefit obtained by B derive directly, according to the applicable legal interpretation, from what is protected?

In turn, point 1.3.1 opens up a series of considerations and further sub-branches that go into the merits of the specific legislation in force in Italy (or in another country), which in turn refer to international conventions.

Now it should be quite clear how it is possible to continue reasoning *ad infinitum* by successive, more or less complex articulations, which we can think of as deviations from a simple ideal straight line. And it is precisely the infinite branching or jaggedness that allows the leap towards fractal figures and fractal geometry. The above iterations, therefore, are to be understood as unlimited and with endless descending ramifications. Thus, to measure or evaluate the distance between any two points, however understood (two laws, two normative entities, or two competing positions A and B as in our case), the length of a piece of the von Koch curve seems very appropriate.

To show a numerical example and, at the same time, the great usefulness, or necessity in this context, of grossone methodology, let us consider the following two points:

$$A = (0,0) \qquad \text{and} \qquad B = \left(\frac{1}{3}, \frac{\sqrt{3}}{9}\right)$$

belonging to all K_n, $n \geq 2$, and so also to K. The distance between A and B along K_2 is 2/3, along K_3 is 8/9 and in general, along K_n is

$$\frac{1}{2} \cdot \left(\frac{4}{3}\right)^{n-1}$$

for all $n \geq 2$. This means that the distance between A and B along the von Koch fractal curve after is an infinite distance. For instance, considering ① steps in the construction (i.e. $K_{①}$), such infinite distance is expressible through the grossone-based system as

$$\frac{1}{2} \cdot \left(\frac{4}{3}\right)^{①-1}.$$

In conclusion, it is evident that the same procedure described above can be applied, at least in theory, in many different situations that arise in the legal field.

4 Future work

It is clear that there is still much to be studied and developed along the lines sketched out in this article. There are many legal contexts to be examined in the future. Furthermore, from a mathematical point of view, the von Koch fractal curve has many variants of different types, which can be considered. For instance, to give a very simple example, one can consider unequal tripartitions of the original interval $[0,1] \times \{0\}$ and constructions with triangles other than the equilateral one. This would mean introducing construction parameters, even variable, that could be calibrated *ad hoc* to best interpret different contexts and situations.

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